Formula Sheet

Integration

$$\int_a^b x^d dx = \frac{x^{d+1}}{d+1}\Big|_a^b = \frac{b^{d+1} - a^{d+1}}{d+1} \quad \text{for } d \neq -1$$

Derivatives

$$f(x) = x^a, \qquad f'(x) = \frac{df}{dx}(x) = ax^{a-1}$$

$$f(x) = \exp(x), \qquad f'(x) = \frac{df}{dx}(x) = \exp(x)$$

$$f(x) = \ln(x), \qquad f'(x) = \frac{df}{dx}(x) = \frac{1}{x}$$

$$f(x) = g(h(x)), \quad u = h(x) \qquad f'(x) = \frac{df}{dx}(x) = \frac{dg}{du}\frac{du}{dx}(x) = g'(u)h'(x) \quad \triangleright \text{Chain rule}$$

$$f(x) = g(x)h(x), \qquad f'(x) = \frac{df}{dx}(x) = g'(x)h(x) + g(x)h'(x) \quad \triangleright \text{Product rule}$$

Probability

Univariate: $\mathbb{P}(X \in \mathcal{E}) \overset{\text{def}}{=} \begin{cases} \sum_{x\in\mathcal{E}} p(x) & \text{if } X \text{ is discrete} \\ \int_{\mathcal{E}} p(x)dx & \text{if } X \text{ is continuous} \end{cases}$

Multivariate: $\mathbb{P}(X \in \mathcal{E}_X, Y \in \mathcal{E}_Y) \overset{\text{def}}{=} \begin{cases} \sum_{x\in\mathcal{E}_x} \sum_{y\in\mathcal{E}_y} p(x,y) & \text{if } X, Y \text{ are discrete} \\ \int_{\mathcal{E}_x} \int_{\mathcal{E}_y} p(x,y)dydx & \text{if } X, Y \text{ are continuous} \\ \int_{\mathcal{E}_x} \sum_{y\in\mathcal{E}_y} p_{Y|X}(y|x)p_X(x)dx & \text{if } X \text{ is continuous, } Y \text{ is discrete} \\ \sum_{x\in\mathcal{E}_x} \int_{\mathcal{E}_y} p_{Y|X}(y|x)p_X(x)dy & \text{if } X \text{ is discrete, } Y \text{ is continuous} \end{cases}$

Marginal pmf: $p_X(x) \overset{\text{def}}{=} \begin{cases} \sum_{y\in\mathcal{Y}} p(x,y) & \text{if } Y \text{ is discrete} \\ \int_{\mathcal{Y}} p(x,y)dy & \text{if } Y \text{ is continuous} \end{cases}$

Marginal: $\mathbb{P}_X(X \in \mathcal{E}_X) \overset{\text{def}}{=} \begin{cases} \sum_{x\in\mathcal{E}_X} p_X(x) & \text{if } X \text{ is discrete} \\ \int_{\mathcal{E}_X} p_X(x)dx & \text{if } X \text{ is continuous} \end{cases}$

Conditional pmf: $p_{Y|X}(y|x) \overset{\text{def}}{=} \frac{p(x,y)}{p_X(x)} \quad \text{such that } p_X(x) > 0$

Conditional: $\mathbb{P}_{Y|X}(Y \in \mathcal{E}_Y | X = x) \overset{\text{def}}{=} \begin{cases} \sum_{y\in\mathcal{E}_Y} p_{Y|X}(y|x) & \text{if } Y \text{ is discrete} \\ \int_{\mathcal{E}_Y} p_{Y|X}(y|x)dy & \text{if } Y \text{ is continuous} \end{cases}$

Product Rule: $p(x,y) = p_{Y|X}(y|x)p_X(x)$

Bayes' Rule: $p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)p_X(x)}{p_Y(y)}$

Independence: $p(x_1, \ldots, x_n) = p_{X_1}(x_1) \cdots p_{X_n}(x_n)$

Distribution	Parameters	pmf or pdf	Expectation and Variance
Bernoulli	$\alpha \in [0,1]$	$p(x) = \alpha^x(1-\alpha)^{1-x},\, x \in \{0,1\}$	$\mathbb{E}[X] = \alpha,\, \text{Var}[X] = \alpha(1-\alpha)$
Discrete Uniform	$n \in \mathbb{N}$	$p(x) = \frac{1}{n},\, x \in \{1,\dots,n\}$	$\mathbb{E}[X] = \frac{n+1}{2},\, \text{Var}[X] = \frac{n^2-1}{12}$
Continuous Uniform	$a, b \in \mathbb{R}, a < b$	$p(x) = \frac{1}{b-a},\, x \in [a,b]$	$\mathbb{E}[X] = \frac{a+b}{2},\, \text{Var}[X] = \frac{(b-a)^2}{12}$
Normal	$\mu \in \mathbb{R}, \sigma^2 > 0$	$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\, x \in \mathbb{R}$	$\mathbb{E}[X] = \mu,\, \text{Var}[X] = \sigma^2$
Laplace	$\mu \in \mathbb{R}, b > 0$	$p(x) = \frac{1}{2b} \exp\left(-\frac{\lvert x-\mu \rvert}{b}\right),\, x \in \mathbb{R}$	$\mathbb{E}[X] = \mu,\, \text{Var}[X] = 2b^2$

Expectation and Variance

Univariate: $\mathbb{E}[X] \stackrel{\text{def}}{=} \begin{cases} \sum_{x \in \mathcal{X}} xp(x) & \text{if } X \text{ is discrete} \\ \int_{\mathcal{X}} xp(x)dx & \text{if } X \text{ is continuous} \end{cases}$

Function: $\mathbb{E}[f(X)] \stackrel{\text{def}}{=} \begin{cases} \sum_{x \in \mathcal{X}} f(x)p(x) & \text{if } X \text{ is discrete} \\ \int_{\mathcal{X}} f(x)p(x)dx & \text{if } X \text{ is continuous} \end{cases}$

Variance: $\text{Var}[X] \stackrel{\text{def}}{=} \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2$

Multivariate: $\mathbb{E}[f(X,Y)] \stackrel{\text{def}}{=} \begin{cases} \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} f(x,y)p(x,y) & \text{if } X \text{ and } Y \text{ are discrete} \\ \int_{\mathcal{X}} \int_{\mathcal{Y}} f(x,y)p(x,y)\,dy\,dx & \text{if } X \text{ and } Y \text{ are continuous} \\ \int_{\mathcal{X}} \sum_{y \in \mathcal{Y}} f(x,y)p_{Y|X}(y|x)p_X(x)\,dx & \text{if } X \text{ is continuous, } Y \text{ is discrete} \\ \sum_{x \in \mathcal{X}} \int_{\mathcal{Y}} f(x,y)p_{Y|X}(y|x)p_X(x)\,dy & \text{if } X \text{ is discrete, } Y \text{ is continuous} \end{cases}$

Conditional: $\mathbb{E}[f(Y)|X=x] \stackrel{\text{def}}{=} \begin{cases} \sum_{y \in \mathcal{Y}} f(y)p_{Y|X}(y|x) & \text{if } Y \text{ is discrete} \\ \int_{\mathcal{Y}} f(y)p_{Y|X}(y|x)\,dy & \text{if } Y \text{ is continuous} \end{cases}$

Expectation and Variance Properties

1. $\mathbb{E}[cX] = c\mathbb{E}[X]$
2. $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
3. $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$
4. $\text{Var}[c] = 0$
5. $\text{Var}[cX] = c^2\text{Var}[X]$.

If X and Y are independent:

6. $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$
7. $\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$.

Estimation

Sample Mean: $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

If X_i are i.i.d.: $\mathbb{E}[\bar{X}] = \mathbb{E}[X_1], \quad \text{Var}\left[\bar{X}\right] = \frac{\text{Var}[X_1]}{n}$