Important Announcements

- assignment 1 is due this Friday (Jan 24)
- review session by TA on Monday

Questions

(1) discrete or cont. r.v.? $X \in \mathcal{X} = \mathbb{R}$ $\quad \mathbb{P}(\{1, 5, 78\}) = 1$

(2) discrete or cont. r.v.? $\mathcal{X} = \mathbb{R}$ $\quad \mathbb{P}(\{1, 5, 78\}) = 0.7$

$\mathbb{P}([-1, 0.5]) = 0.3$

(3) what is the co-domain of any pdf?

$p : \mathcal{X} \to [0, \infty)$ typo in notes

(4) $\mathcal{X}$ countable or uncountable set?

(5) pdf of cont. Uniform(a, b)

(6) pdf/pmf of cond. distribution

$$p_{Y|X}(y|x) = \frac{p(x,y)}{p(x)}$$

(7) Bayes' rule $p(x|y) = \dfrac{p(y|x)p(x)}{p(y)}$

(8) product rule for 3-dim. r.v. $p(x_1, x_2, x_3) =$

$p(x_3|x_2, x_1)p(x_2|x_1)p(x_1)$ $\quad p(x_1, x_2) = p(x_1|x_2)p(x_2)$

(g) What is the difference between

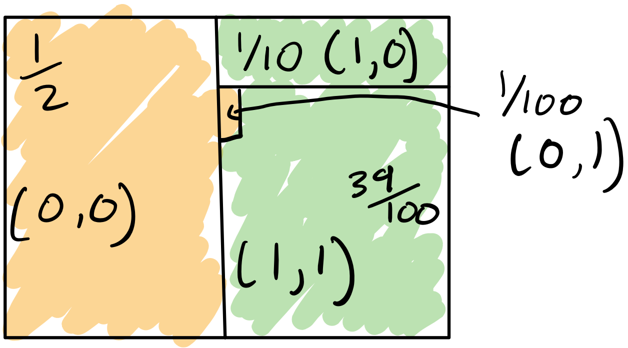
- joint
- marginal
- conditional

distributions?

Arthritis young

$= P_{Y|X}(1|0)$

product rule from above
$p(x,y) = p(x|y)\,p(y)$

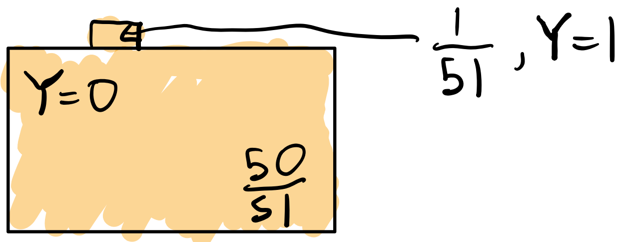


$$= \frac{P_{X,Y}(0,1)}{P_X(0)}$$

$$= \frac{p(0,1)}{p(0,1)+p(0,0)}$$

condition on being young

$$= \frac{1/100}{1/100 + 1/2}$$



$$= \frac{1/100}{51/100} = \frac{1}{51}$$

Ex: Probability of being young given I have arthritis

$$\mathbb{P}(X=0 \mid Y=1) = P_{X|Y}(0|1)$$

Bayes' Rule

$$\overset{\text{Bayes' Rule}}{=} \frac{P_{Y|X}(1|0)\,P_X(0)}{P_Y(1)}$$

$$= \frac{\frac{1}{51}\ \frac{51}{100}}{40/100} = \frac{1}{40}$$

Independence: Changing the value of one r.v. doesn't affect the probability of another r.v.

$\boxed{\text{r.v. } X, Y \text{ are independent if: } \quad p(x,y) = p_X(x)\,p_Y(y)}$

Since $p(x,y) = p(x|y)p(y) = p(x)p(y|x) = p(x)p(y)$

$\boxed{\text{independence implies: } p(x|y) = p(x), \; p(y|x) = p(y)}$

More generally:

$X_1, X_2, \ldots, X_d$ are independent if: $p(x_1, \ldots, x_d) = p(x_1) \cdots p(x_d)$

Similarly for distributions:

$\boxed{\text{r.v. } X, Y \text{ are independent if: } \mathbb{P}(X \in E_X, Y \in E_Y) = \mathbb{P}(X \in E_X)\,\mathbb{P}(Y \in E_Y)}$

Ex: X, Y are <u>not</u> independent for Arthritis ex

$$p(0,1) = \frac{1}{100} \neq p_X(0)\,p_Y(1) = \frac{51}{100}\,\frac{40}{100} = 0.204$$

Ex: $X_1, X_2 \in \{0,1\}$ are flips of two different fair coins

$$p(x_1, x_2) = \frac{1}{4} \quad \text{for all } x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2$$

$$p_{X_1}(x_1)\,p_{X_2}(x_2) = \frac{1}{2}\,\frac{1}{2} = \frac{1}{4}$$

		X_2: H	T
X_1	H	1/4	1/4
	T	1/4	1/4

What happens when $Z=(X,Y)$ with Y discrete and X continuous?

$p: \mathcal{X} \times \mathcal{Y} \Rightarrow ?$ pmf or pdf? Ans: neither

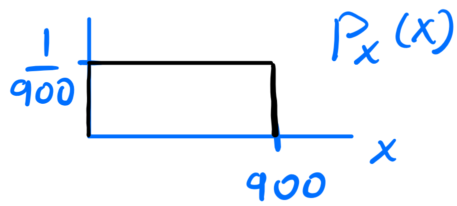
Instead we will write $p(x,y)$ in terms of a marginal pdf for X and a conditional pmf for $Y|X$

$$p(x,y) = p_X(x)\, p_{Y|X}(y|x)$$ product rule

where $p_{Y|X=x}: \mathcal{Y} \Rightarrow [0,1]$ is a pmf
$p_X(x): \mathcal{X} \Rightarrow [0,\infty)$ is a pdf

Ex: $X \in \mathcal{X} = [0,900]$, $Y \in \mathcal{Y} \in \{0,1\}$ Barolo

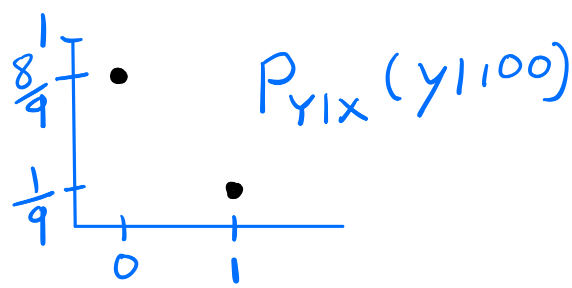
pdf: $p_X = \text{Uniform}(0,900)$
$= \frac{1}{900}$



$P_{Y|X=x} = \text{Bernoilli}\left(\frac{x}{900}\right)$

pmf: $P_{Y|X}(y|x) = \begin{cases} \frac{x}{900} & \text{if } y=1 \\ 1-\frac{x}{900} & \text{if } y=0 \end{cases}$

Defn of Bernoilli$\left(\frac{x}{900}\right)$



$$\mathbb{P}(X \in [0,50], Y=1) = \int_0^{50} \left(\sum_{y \in \{1\}} p(y,x) \right) dx$$

$$= \int_0^{50} \left(\sum_{y \in \{1\}} P_{Y|X}(y|x)\, p_X(x) \right) dx$$

$$= \int_0^{50} p_{Y|X}(1|x)\, p_X(x)\, dx$$

$$= \int_0^{50} \frac{x}{900} \frac{1}{900}\, dx$$

$$= \frac{1}{810000} \left. \frac{x^2}{2} \right|_0^{50}$$

$$= \frac{1}{810000} \frac{2500}{2}$$

$$= \frac{1}{648}$$

0.154%

Representing Random Features, Labels, and Datasets

Random variables:

$D = (Z_1, Z_2, \ldots, Z_n) \in \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_n = \mathcal{Z}^n$ since $\mathcal{Z} = \mathcal{Z}_1 = \cdots = \mathcal{Z}_n$

$Z_i = (\vec{X}_i, Y_i) \in \mathcal{X} \times \mathcal{Y} = \mathcal{Z}$ each Z_i is a feature-label pair

$\vec{X}_i = (X_{i,1}, \ldots, X_{i,d})^T \in \mathbb{R}^d = \mathcal{X}$ $\vec{X}_i$ is a feature vector

Distributions:

$\mathbb{P}_D$: distribution for D, $\mathbb{P}_{Z_i}$: marginal distribution for Z_i

assumptions:

1. $(\vec{X}_i, Y_i) = Z_i$ are independent for all $i \in \{1, \ldots, n\}$
2. $\mathbb{P}_{Z_1} = \mathbb{P}_{Z_2} = \cdots = \mathbb{P}_{Z_n} = \mathbb{P}_Z$ all Z_i have the same distribution

"$(\vec{X}_i, Y_i)$ are independent and identically distributed (i.i.d)"

$$\mathbb{P}_D(Z_1 \in E_1, \ldots, Z_n \in E_n) = \mathbb{P}_{Z_1}(Z_1 \in E_1) \cdots \mathbb{P}_{Z_n}(Z_n \in E_n)$$
$$= \mathbb{P}_Z(Z_1 \in E_1) \cdots \mathbb{P}_Z(Z_n \in E_n)$$

Equivalently:

$D = ((\vec{X}_1, Y_1), \ldots, (\vec{X}_n, Y_n)) \in (\mathcal{X} \times \mathcal{Y})^n$

where $(\vec{X}_i, Y_i) \sim \mathbb{P}_{\vec{X},Y}$ are idependent for all $i \in \{1, \ldots, n\}$

"sampled/distributed according to"

D contains n independent samples of $(\vec{X}_i, Y_i)$ "feature-label" pairs all coming from the same distribution $\mathbb{P}_{\vec{X},Y}$

Functions of Random Variables

A function of a r.v. is a r.v.

Ex: (X is a fair six-sided dice)

$X \in \{1,2,3,4,5,6\} = \mathcal{X}$ with $p(x) = \frac{1}{6}$

$f(X) = X^2 \in \underbrace{\{1^2, 2^2, 3^2, 4^2, 5^2, 6^2\}}_{\text{outcome space for } f(X)} = \mathcal{Y}$ is a r.v.

Notice $f: \mathcal{X} \to \mathcal{Y}$

Sometimes we give the r.v. a new symbol

$Y = f(X) = X^2$

(pmf) $P_Y(y) = P_{f(X)}(y) = \frac{1}{6}$ where $y \in \{1, 2^2, 3^2, 4^2, 5^2, 6^2\}$

In this case $p_Y(x^2) = p(x)$ where $x \in \mathcal{X}$

ex: $p_Y(9) = p_Y(3^2) = p(3) = \frac{1}{6}$

Ex: (X is the payout from a slot machine)

$X \in [-10,10]$ with $p(x)=\frac{1}{20}$, $\mathbb{P}=$ Uniform$(-10,10)$

$Y = f(X) = X^2 \in [0,100] = \mathcal{Y}$

$p_Y(y) = \frac{1}{20\sqrt{y}}$ much more complicated

In general p_Y is complicated and we will not need to know how to calculate it

The Predictor and Learner are functions of r.v.

Ex: (Predictor)

$\vec{X} = (X_1, X_2)^T \in \mathbb{R}^2 = \mathcal{X}$ with $\mathbb{P}_{\vec{X}}$

predictor: $f: \mathcal{X} \to \mathcal{Y}$ where $\mathcal{Y} = \mathbb{R}$

$f(\vec{X}) = 3 + 6X_1 + 2.5X_2$ is a r.v. with values in $\mathcal{Y}$

and has some distribution $\mathbb{P}_{f(\vec{X})}$

Ex: (Learner)

$D = ((\vec{X}_1, Y_1), \ldots, (\vec{X}_n, Y_n)) \in (\mathcal{X} \times \mathcal{Y})^n$ with $\mathbb{P}_D$

Learner: $\mathcal{A}: (\mathcal{X} \times \mathcal{Y})^n \Rightarrow \{f \mid f: \mathcal{X} \Rightarrow \mathcal{Y}\} = \mathcal{F}$

$\mathcal{A}(D) = f$ is a r.v. with values in $\mathcal{F}$

and has some distribution $\mathbb{P}_{\mathcal{A}(D)}$

example

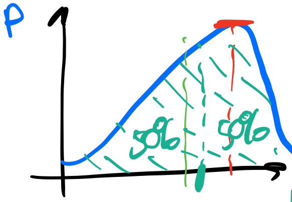
if $D = ((7, 6), (12, 2.5))$ where $n = 2$, $\mathcal{X} = \mathbb{R}$, $\mathcal{Y} = \mathbb{R}$

then f_D can be $f(x) = 2.5 + 6x$

This means we can talk about things like:

- What is the probability the Predictor $f(\vec{x})$ outputs some value y
- What is the probability the Learner $\mathcal{A}(D)$ outputs some predictor f

Expectation and Variance



Expected Value of a r.v.: average value of the r.v. if you sample from its distribution infinitely many times.

The r.v. must take values in $\mathbb{R}$.

It is not always the value we expect to see most frequently (that is the mode)

$X \in \mathcal{X}$ is a r.v. with pmf or pdf p

$$\boxed{\mathbb{E}[X] \stackrel{\text{def}}{=} \begin{cases} \sum_{x \in \mathcal{X}} x\, p(x) & \text{if } X \text{ is discrete} \\ \int_{\mathcal{X}} x\, p(x)\, dx & \text{if } X \text{ is continuous} \end{cases}}$$

Ex: (fair six-sided dice)

$X \in \{1,2,3,4,5,6\} = \mathcal{X}$ and $\mathbb{P} = \text{Uniform}(n=6)$

thus $p(x) = \frac{1}{6}$

$$\mathbb{E}[X] = \sum_{x \in \mathcal{X}} x\, p(x) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

$$= 3.5$$

This is not an number you can roll on a dice!

Ex: (Unfair coin)

$X \in \{0, 1\}$ and $\mathbb{P} = \text{Bernoulli}(\alpha)$

thus $p(1) = \alpha, \; p(0) = 1 - \alpha$

$$\mathbb{E}[X] = \sum_{x \in \mathcal{X}} x \, p(x) = 0 \cdot (1-\alpha) + 1 \cdot \alpha$$
$$= \alpha$$

This is <u>not</u> a result of a coin flip (unless $\alpha = 1$ or $\alpha = 0$)

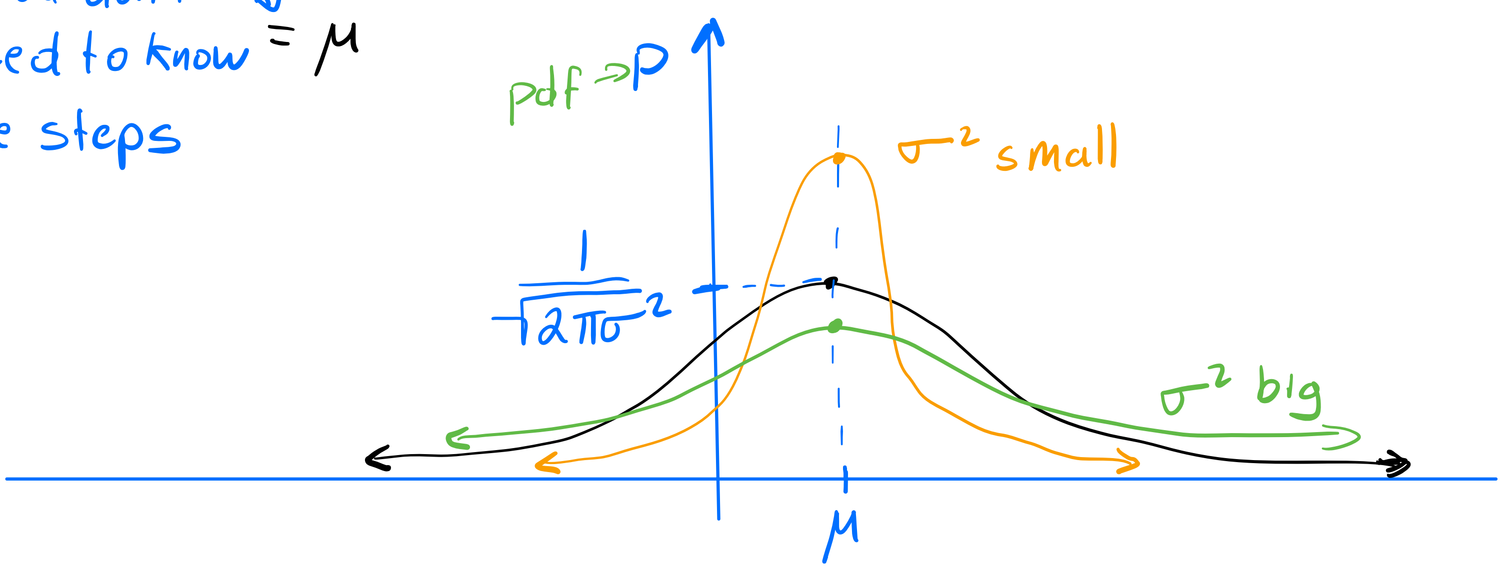
Ex: (Normal distribution)

$X \in \mathbb{R} = \mathcal{X}$ and $\mathbb{P} = \mathcal{N}(\mu, \sigma^2)$

thus $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$

$$\mathbb{E}[X] = \int_{\mathcal{X}} x \, \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx$$
$$= \mu$$

You don't need to know the steps



Expected value of functions of r.v.:

$X \in \mathcal{X}$ is a r.v. with pmf or pdf p

The function $f : \mathcal{X} \to \mathcal{Y}$ must have $\mathcal{Y} = \mathbb{R}$

$$\boxed{\mathbb{E}[f(X)] \overset{\text{def}}{=} \begin{cases} \sum_{x \in \mathcal{X}} f(x) p(x) & \text{if } X \text{ is discrete} \\ \int_{\mathcal{X}} f(x) p(x)\, dx & \text{if } X \text{ is continuous} \end{cases}}$$

Ex: (X is the payout from a slot machine)

$X \in [-10, 10]$ with $p(x) = \frac{1}{20}$, $\mathbb{P} = \text{Uniform}(-10, 10)$

$Y = f(X) = X^2 \in [0, 100] = \mathcal{Y}$

$p_Y(y) = \frac{1}{20\sqrt{y}}$ much more complicated

$$\mathbb{E}[f(X)] = \int_{\mathcal{X}} f(x) p(x)\, dx = \int_{-10}^{10} x^2 \frac{1}{20}\, dx$$

$$= \left. \frac{x^3}{3} \cdot \frac{1}{20} \right|_{-10}^{10}$$

$$= \left(\frac{1000}{3} - \left(\frac{-1000}{3} \right) \right) \cdot \frac{1}{20}$$

$$= \frac{2000}{60} = 33.333$$

It turns out

$$\mathbb{E}[f(X)] = \mathbb{E}[Y] = \int_{y} y \, p_Y(y) \, dy$$

$$= 33.333$$

Usually we don't know $p_Y = p_{f(X)}$

So we work with p

Variance of a r.v.: How much the r.v. varies from its expected value on average

$X \in \mathcal{X}$ is a r.v. with pmf or pdf p

$$\boxed{\mathrm{Var}[X] \overset{\text{def}}{=} \mathbb{E}\Big[\underbrace{\big(X - \mathbb{E}[X]\big)^2}\Big] = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2}$$

this is just a function of the r.v. X

Ex: (Unfair coin)

$X \in \{0, 1\}$ and $\mathbb{P} = \text{Bernoulli}(\alpha)$

thus $p(1) = \alpha, p(0) = 1 - \alpha$

$$\mathbb{E}[X] = \sum_{x \in \mathcal{X}} x\, p(x) = 0 \cdot (1-\alpha) + 1 \cdot \alpha$$
$$= \alpha$$

$$\mathrm{Var}[X] = \mathbb{E}\left[\left(X - \mathbb{E}[X]\right)^2\right]$$
$$= \sum_{x \in \mathcal{X}} (x - \mathbb{E}[X])^2 p(x)$$
$$= (0-\alpha)^2 \cdot (1-\alpha) + (1-\alpha)^2 \cdot \alpha$$
$$= \alpha^2 - \alpha^3 + \alpha - 2\alpha^2 + \alpha^3$$
$$= \alpha - \alpha^2$$
$$= \alpha(1-\alpha)$$

or $\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$$= \sum_{x \in \mathcal{X}} x^2 p(x) - \alpha^2$$
$$= 0^2 \cdot (1-\alpha) + 1^2 \cdot \alpha - \alpha^2$$
$$= \alpha(1-\alpha)$$

<u>Ex</u>: (Normal distribution)

$X \in \mathbb{R} = \mathcal{X}$ and $\mathbb{P} = \mathcal{N}(\mu, \sigma^2)$

thus $p(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\dfrac{1}{2\sigma^2}(x-\mu)^2\right)$

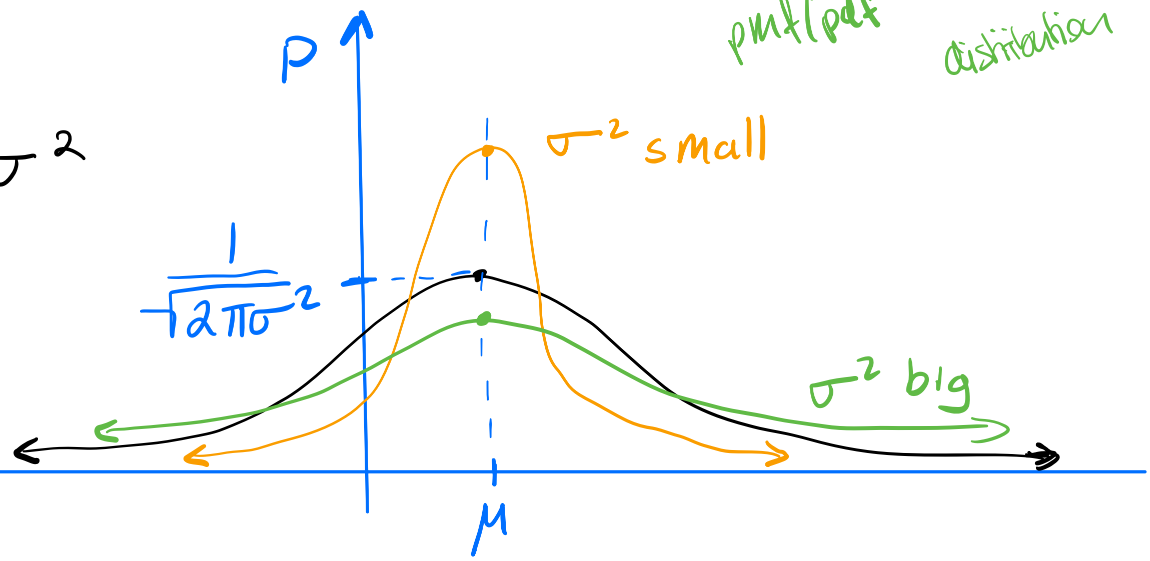
$$\mathbb{E}[X] = \int_{\mathcal{X}} x \, \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx$$

You don't need to know the steps $= \mu$

$$\mathrm{Var}[X] = \sigma^2$$

$p(x)$ $\mathbb{P}(X \in E)$

pmf/pdf distribution



Multivariate Expected Value:

$$Z = (X, Y) \in \mathcal{X} \times \mathcal{Y} = \mathcal{Z}$$ is a r.v.

$$f : \mathcal{X} \times \mathcal{Y} \Rightarrow \mathbb{R}$$

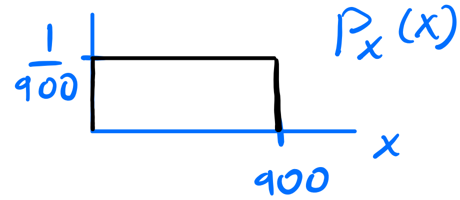
$$\mathbb{E}[f(X,Y)] = \begin{cases} \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} f(x,y) p(x,y) & \text{if } X, Y \text{ are discrete} \\ \int_{\mathcal{X}} \int_{\mathcal{Y}} f(x,y) p(x,y) \, dy \, dx & \text{if } X, Y \text{ are continuous} \\ \int_{\mathcal{X}} \left(\sum_{y \in \mathcal{Y}} f(x,y) p(y|x) \right) p(x) \, dx & \text{if } Y \text{ is discrete and } X \text{ is continuous} \\ \sum_{x \in \mathcal{X}} \left(\int_{\mathcal{Y}} f(x,y) p(y|x) \, dy \right) p(x) & \text{if } Y \text{ is continuous and } X \text{ is discrete} \end{cases}$$

you can always use:

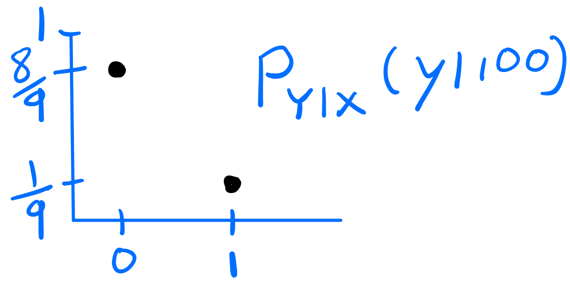
$$p(x,y) = p(y|x)\,p(x) = p(x|y)\,p(x)$$

$\underline{\underline{Ex}}$: $X \in \mathcal{X} = [0, 900]$, $Y \in \mathcal{Y} \in \{0, 1\}$ Barolo

pdf: $p_X = \text{Uniform}(0, 900)$
$= \frac{1}{900}$



$p_{Y|X=x} = \text{Bernoilli}\left(\frac{x}{900}\right)$



pmf: $p_{Y|X}(y|x) = \begin{cases} \frac{x}{900} & \text{if } y = 1 \\ 1 - \frac{x}{900} & \text{if } y = 0 \end{cases}$

Defn of Bernoilli$\left(\frac{x}{900}\right)$

$$f(X,Y) = \left(\frac{X}{900} - Y\right)^2$$

$$\mathbb{E}[f(X,Y)] = \int_{\mathcal{X}} \left(\sum_{y \in \mathcal{Y}} f(x,y)\, p(y|x) \right) p(x)\, dx$$

$$= \int_0^{900} \left(\sum_{y \in \{0,1\}} \left(\frac{x}{900} - y\right)^2 p(y|x) \right) p(x)\, dx$$

$$= \int_0^{900} \left(\left(\frac{x}{900} - 0\right)^2 \left(1 - \frac{x}{900}\right) + \left(\frac{x}{900} - 1\right)^2 \left(\frac{x}{900}\right) \right) \frac{1}{900}\, dx$$

$$= \frac{1}{900} \int_0^{900} \frac{x}{900} \left(1 - \frac{x}{900}\right) dx$$

$$= \frac{1}{900} \left(\frac{x^2}{1800} \Big|_0^{900} - \frac{x^3}{3 \cdot 900^2} \Big|_0^{900} \right)$$

$$= \frac{1}{6}$$

Conditional Expected Value:

$(X, Y) \in \mathcal{X} \times \mathcal{Y}$ is a r.v.

$p = p_{Y|X}$ is a conditional pmf or pdf

$f : \mathcal{Y} \to \mathbb{R}$

$$\boxed{\mathbb{E}[f(Y) \mid X = x] = \begin{cases} \sum_{y \in \mathcal{Y}} f(y) p(y|x) & \text{if } Y \text{ is discrete} \\ \int_{\mathcal{Y}} f(y) p(y|x) & \text{if } Y \text{ is continuous} \end{cases}}$$

Useful Properties

Let X, Y be r.v. and $c \in \mathbb{R}$ be a constant

1. $\mathbb{E}[cX] = c\,\mathbb{E}[X]$
2. $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
3. $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]]$
4. $Var[c] = 0$
5. $Var[cX] = c^2\,Var[X]$

will be on the formula sheet for midterms

If X and Y are independent:

6. $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$
7. $Var[X+Y] = Var[X] + Var[Y]$