Estimating $P_{Y|\vec{X}}$

$$D = ((\vec{X}_1, Y_1), \dots, (\vec{X}_n, Y_n)) \in (\mathcal{X} \times \mathcal{Y})^n, \ \mathbb{P}_D, \ p_D$$

$(\vec{X}_i, Y_i)$ are i.i.d with $\mathbb{P}_{\vec{X},Y}$ and $p_{\vec{X},Y}$

$$p_{\vec{x},y}(\vec{x}, y) \overset{\text{product rule}}{=} p_{y|\vec{x}}(y|\vec{x})\, p_{\vec{x}}(\vec{x})$$

Assume $Y_i | \vec{X}_i = \vec{x}_i \sim \mathcal{N}(\vec{x}_i^T \vec{w}^*, 1)$

$$p_{y|\vec{x}}(y_i | \vec{x}_i) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y_i - \vec{x}_i^T \vec{w}^*)^2}{2}\right)$$

calculating $\vec{w}_{MLE}$

$$\vec{w}_{MLE} = \underset{\vec{w} \in \mathbb{R}^{d+1}}{\arg\max}\ p_D(D|\vec{w})$$

$$= \underset{\vec{w} \in \mathbb{R}^{d+1}}{\arg\max} \prod_{i=1}^{n} p(\vec{x}_i, y_i | \vec{w})$$

$$= \underset{\vec{w} \in \mathbb{R}^{d+1}}{\arg\min} -\log\left(\prod_{i=1}^{n} p(\vec{x}_i, y_i | \vec{w})\right)$$

$$= \underset{\vec{w} \in \mathbb{R}^{d+1}}{\arg\min} \sum_{i=1}^{n} -\log\left(p(\vec{x}_i, y_i | \vec{w})\right)$$

$$= \underset{\vec{w} \in \mathbb{R}^{d+1}}{\arg\min} \sum_{i=1}^{n} -\log\left(p(y_i | \vec{x}_i, \vec{w})\, p(\vec{x}_i)\right)$$

$$= \underset{\vec{w} \in \mathbb{R}^{d+1}}{\arg\min} - \sum_{i=1}^{n} \left[\log\left(p(y_i | \vec{x}_i, \vec{w}) \right) + \log\left(p(\vec{x}_i) \right) \right]$$

$$= \underset{\vec{w} \in \mathbb{R}^{d+1}}{\arg\min} - \sum_{i=1}^{n} \log\left(p(y_i | \vec{x}_i, \vec{w}) \right)$$

$$= \underset{\vec{w} \in \mathbb{R}^{d+1}}{\arg\min} - \sum_{i=1}^{n} \log\left(\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y - \vec{x}^T \vec{w})^2}{2} \right) \right)$$

$$= \underset{\vec{w} \in \mathbb{R}^{d+1}}{\arg\min} - \sum_{i=1}^{n} \left[\log\left(\frac{1}{\sqrt{2\pi}} \right) + \log\left(\exp\left(-\frac{(y - \vec{x}^T \vec{w})^2}{2} \right) \right) \right]$$

$$= \underset{\vec{w} \in \mathbb{R}^{d+1}}{\arg\min} \sum_{i=1}^{n} \frac{(y_i - \vec{x}_i^T \vec{w})^2}{2}$$

$$\Rightarrow \quad \vec{w}_{MLE} = \underset{\vec{w} \in \mathbb{R}^{d+1}}{\arg\min} \sum_{i=1}^{n} \frac{(y_i - \vec{x}_i^T \vec{w})^2}{2}$$

$$p_{MLE}(y|\vec{x}) = p(y|\vec{x}, \vec{w}_{MLE}) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y_i - \vec{x}_i^T \vec{w}_{MLE})^2}{2}\right)$$

$$\approx p_{Y|\vec{X}}(y|\vec{x})$$

$$f_{Bayes}(\vec{x}) = \mathbb{E}[Y | \vec{X} = \vec{x}]$$

$$= \int_y y \, p_{Y|\vec{X}}(y|\vec{x}) \, dy$$

$$\approx \int_y y \, p(y|\vec{x}, \vec{w}_{MLE}) \, dy$$

$$= \mathbb{E}[Y' | \vec{X} = \vec{x}] \text{ where } Y' | \vec{X} = \vec{x} \sim \mathcal{N}(\vec{x}^T \vec{w}_{MLE}, 1)$$

$$= \vec{x}^T \vec{w}_{MLE}$$

$$= \vec{x}^T \hat{\vec{w}}$$

$$= \hat{f}$$