Important Announcements and Notes

- Midterms changed to 75 min (12:30pm - 1:45pm)
- These Lecture Notes are posted on the course website in the "Schedule tab"
- Update course notes will be posted by Fri night

- We will use tuple instead of ordered set
- Review why having duplicates in D is important
- We have not covered the objective or where the data comes from yet
- Review $\{f \mid f : \mathcal{X} \Rightarrow \mathcal{Y}\}$
- $\vec{x}_1 = (x_{1,1}, \ldots, x_{1,d})^T$
- Contable vs. Uncountable infinity

Math Review

- Sets and Vectors
- functions

Motivation

Supervised Learning: Learning from a randomly sampled batch of labeled data

# of rooms	price
2	200
4	590
3	350
7	970

Predictor function f
input: # of rooms
output: price

Experience $\xrightarrow{\text{Learning}}$ knowledge

formalize

Programs/Algorithms implement functions

Dataset (tuple of tuples) $\xrightarrow{\text{Learner (function)}}$ Predictor = Model = Hypothesis (function)

$(x_{1,1}, \ldots, x_{1,d})^T$

$$\boxed{\begin{gathered} D = ((\vec{x}_1, y_1), (\vec{x}_2, y_2), \ldots, (\vec{x}_n, y_n)) \\ D \in (\mathcal{X} \times \mathcal{Y})^n \end{gathered}}$$

D n feature-label pairs

$\mathcal{X}$ set of features

$\mathcal{Y}$ set of labels/targets

Ex: $\mathcal{X} = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, $\mathcal{Y} = \mathbb{R}$

features	Label
$\vec{x}_1 = (x_{1,1}, x_{1,2})$	y_1
$\vec{x}_2 = (x_{2,1}, x_{2,2})$	y_2
$\vdots$	$\vdots$
$\vec{x}_n = (x_{n,1}, x_{n,2})$	y_n

$\vec{x}_i \in \mathcal{X}$, $y_i \in \mathcal{Y}$, $i \in \{1, \ldots, n\}$

$$\boxed{f : \mathcal{X} \to \mathcal{Y}}$$

f a function from features to labels

Ex $f(x) = 2x + 1$, $\mathcal{X} = \mathbb{R}$

$$\boxed{\mathcal{A} : (\mathcal{X} \times \mathcal{Y})^n \to \{f \mid f : \mathcal{X} \to \mathcal{Y}\}}$$

$\mathcal{A}$ a function from datasets to predictors

Ex: $\mathcal{A}(D) = f$ where $f : \mathcal{X} \to \mathcal{Y}$

$$f(x) = \begin{cases} y_i & \text{if } x = x_i \text{ for some } i \in \{1, \ldots, n\} \\ 0 & \text{otherwise} \end{cases}$$

Sets

Set: a collection of distinct and unordered objects

Ex: $\{0,1,2\} = \{0,1,2,2\}$, $\{cat, dog\} = \{dog, cat\}$

$\mathbb{N} = \{0,1,2,\dots\}$ natural numbers

$\mathbb{R}$ real numbers, $\emptyset$ empty set

Variables as sets: Ω, X, Y, Z

Ex: $X = \{0,1,2\}$, $\Omega = \{cat, dog\}$

Cardinality: size of the set

Ex: $|X| = 3$, $|\Omega| = 2$, $|\emptyset| = 0$, $|\mathbb{N}|$, $|\mathbb{R}|$

Fun fact
$|\mathbb{N}| \neq |\mathbb{R}|$

Countably Infinite Set: If you can list all the elements in the set

Ex: $|\mathbb{N}|$ = countably infinite, $\mathbb{N} = \{0,1,2,3,\dots\}$

$\{x \in \mathbb{N} \mid x \text{ is even}\} = \{0,2,4,6,\dots\}$

Uncountably Infinite Set: Not countably infinite

Ex: $|\mathbb{R}|$ = uncountably infinite, $\mathbb{R} \neq \{0, \overset{0.00001}{\vee} 0.0001, 0.0002, \dots\}$

$|[0,900]|$ = uncountably infinite

Element of and Subset of: $\in, \notin, \subset, \not\subset$

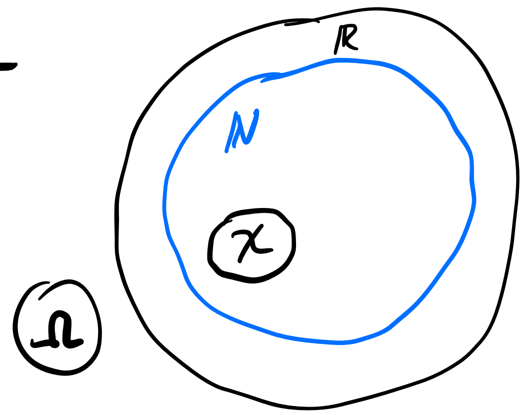
Ex: $cat \in \Omega$ cat is an element of Ω

$cat \notin \mathcal{X} = \{0,1,2\}$

$0 \in \mathbb{R}, 1 \in \mathbb{R}, -2 \in \mathbb{R}, \frac{1}{2} \in \mathbb{R}, 0.23 \in \mathbb{R}, \pi \in \mathbb{R}, \infty \notin \mathbb{R}$

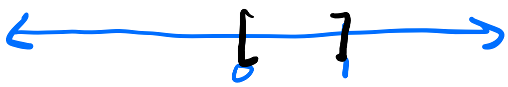
$\mathcal{X} \subset \mathbb{N}, \mathcal{X} \subset \mathbb{R}, \mathbb{N} \subset \mathbb{R}, \mathcal{X} \not\subset \Omega$

$\mathcal{X} \subset \mathcal{X}, \Omega \subset \Omega$



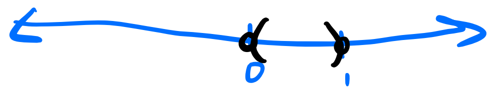
Intervals: Continuous subset of $\mathbb{R}$

Closed: $[0,1] \subset \mathbb{R}$

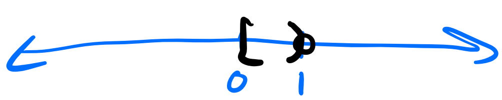


$[0,1] \not\subset \mathbb{N}$

open: $(0,1) \subset \mathbb{R}, 0 \notin (0,1)$



half-open: $[0,1) \subset \mathbb{R}, 1 \notin [0,1)$



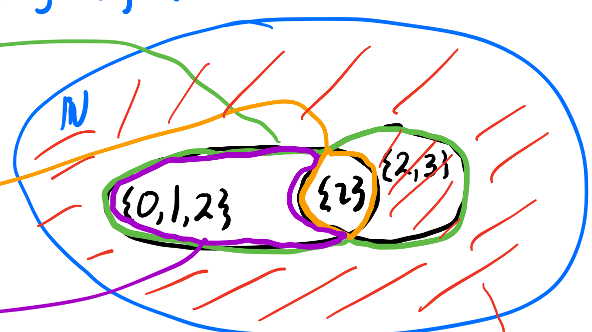
$[0,\infty)$

Unions, Intersections, Set Difference: $\cup, \cap, \setminus$

$\{0,1,2\} \cup \{2,3\} = \{0,1,2,3\}$

$\{0,1,2\} \cap \{2,3\} = \{2\}$

$\{0,1,2\} \setminus \{2,3\} = \{0,1\}$



Compliment: set^c Need to define a universal set $\mathcal{U}$

Ex: $\mathcal{U}=\mathbb{N}$, $\mathcal{X}^c=\{0,1,2\}^c=\mathcal{U}\setminus\mathcal{X}=\{3,4,5,\dots\}$

"such that"

Set Builder Notation: $\{\text{element} \mid \text{property}\}$

Ex: $\{x\in\mathbb{N} \mid x\leq 2\}=\{0,1,2\}$

$\{x\in\mathbb{N} \mid x \text{ is even}\}=\{0,2,4,\dots\}$

$\{x\in\mathbb{N} \mid x \text{ is prime}\}=\{3,5,7,11,\dots\}$

$\{x\in\mathbb{N} \mid x\in\{0,1,2\} \text{ and } x\notin\{2,3\}\}=\{0,1,2\}\setminus\{2,3\}$

Power set: $\mathcal{P}(\mathcal{Z})=\{\mathcal{S} \mid \mathcal{S}\subset\mathcal{Z}\}$ the set of all subsets of $\mathcal{Z}$

Ex: $\mathcal{P}(\{0,1\})=\{\emptyset,\{0\},\{1\},\{0,1\}\}$

Tuple: a collection of ordered objects with duplicates allowed

Ex: $(0,1,2)\neq(0,1,2,2)$, $(\text{cat},\text{dog})\neq(\text{dog},\text{cat})$ This is not an interval!

only variable and element of properties apply

$\Omega'=(0,1,2)$, $\mathcal{X}'=(\text{cat},\text{dog})$, $2\in\Omega'$, $2\notin\mathcal{X}'$

Invalid: $\Omega'\subset\mathcal{X}'$, $\Omega'\cup\mathcal{X}'$, $\Omega'\cap\mathcal{X}'$, ...

Cartesian products (creating tuples): set $\times$ set

Ex: $\Omega\times\mathcal{X}=\{\text{cat},\text{dog}\}\times\{0,1,2\}\neq\mathcal{X}\times\Omega$

$=\{(\text{cat},0),(\text{cat},1),(\text{cat},2),(\text{dog},0),(\text{dog},1),(\text{dog},2)\}$

$=\{(a,b) \mid a\in\Omega, b\in\mathcal{Z}\}$

Ex: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a,b) \mid a \in \mathbb{R},\ b \in \mathbb{R}\}$

$(0,2) \in \mathbb{R}^2$, $(-\frac{1}{10}, \pi) \in \mathbb{R}^2$

Ex: $[0,1]^2 = [0,1] \times [0,1] = \{(a,b) \mid a \in [0,1], b \in [0,1]\}$

Ex: $\mathbb{R}^3 = \{(a,b,c) \mid a \in \mathbb{R}, b \in \mathbb{R}, c \in \mathbb{R}\}$

Ex: $\mathcal{X} = \mathbb{R}^2$, $\mathcal{Y} = \mathbb{R}$

$\mathcal{X} \times \mathcal{Y} = \{(x,y) \mid x \in \mathcal{X}, y \in \mathcal{Y}\}$

$\in \mathbb{R}^3 \neq \mathbb{R}^2 = \mathcal{X}$

Notice how order is important here

$((2,2),200) \in \mathcal{X} \times \mathcal{Y}$, $((2,12,1),200) \notin \mathcal{X} \times \mathcal{Y}$

$\mathcal{X} \times \mathcal{Y} = (\mathbb{R}^2) \times \mathbb{R} \neq \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$, $(2,2,200) \in \mathbb{R}^3$

Ex: $(\mathcal{X} \times \mathcal{Y})^n = \underset{1}{(\mathcal{X} \times \mathcal{Y})} \times \underset{2}{(\mathcal{X} \times \mathcal{Y})} \times \underset{\dots}{\dots} \times \underset{n}{(\mathcal{X} \times \mathcal{Y})}$ Duplicates are allowed

$D = (((2,2),200), ((4,10),450), \dots, ((2,2),200))$

$\in (\mathcal{X} \times \mathcal{Y})^n$

Vectors

Motivation: Can model relationships between features and targets

Ex: targets are a linear function of the features (i.e. $y = \vec{x}^T \vec{w}$)

Vector space: A set of tuples that we can add together any elements and multiply any element by a real number (i.e. scalar)

Ex: $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \dots$ why?

Ans ($\mathbb{R}^2$): $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$, $c \in \mathbb{R}$

adding: $\vec{x} + \vec{y} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \in \mathbb{R}^2$

multiplying by a scalar: $c\vec{x} = c\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \end{pmatrix} \in \mathbb{R}^2$

$\Omega = \{dog, cat\}$ is not a vector space why?

Ans: what does dog + cat mean?

Vector: An element of a Vector space (written as a column)

Ex: $\vec{x} = \begin{pmatrix} 1 \\ 2.5 \end{pmatrix} \in \mathbb{R}^2$, $\vec{w} = \begin{pmatrix} -12 \\ 10 \end{pmatrix} \in \mathbb{R}^2$, $y = 3 \in \mathbb{R}$

Transpose: Changes a column vector to a row vector and vice versa

Ex: $\vec{x}^T = (1, 2.5) \notin \mathbb{R}^2$, $\vec{w}^T = (-12, 10)$, $y^T = 3$

Row vectors belong to a more complicated space so we will not mention it, and instead write the Vector space that the column vector belongs to. Ex: $\vec{x} \in \mathbb{R}^2$

Dot Product: A way to multiply two vectors

Ex: $\vec{w}^T\vec{x} = \vec{x}^T\vec{w} = (x_1, x_2)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = x_1w_1 + x_2w_2 = -12 + 25 = 13$

Matrix: Multiple row vectors vertically stacked

Ex: $M = \begin{bmatrix} \vec{x}^T \\ \vec{w}^T \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ w_1 & w_2 \end{bmatrix}$, $M' = \begin{bmatrix} 1 & 2 \\ 1.5 & 3 \\ 0 & -2 \end{bmatrix}$

$\vec{x}^T = (1, 2.5)$

$\vec{x} = \begin{pmatrix} 1 \\ 2.5 \end{pmatrix}$

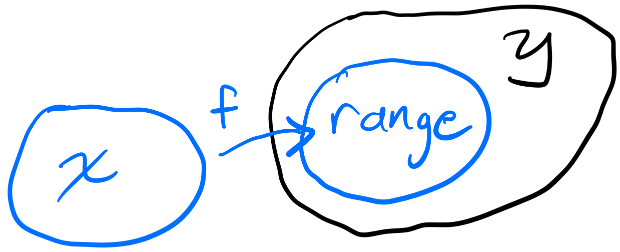
$\vec{x} = (1, 2.5)^T$

Matrix vector multiplication

Ex: $M\vec{x} = \begin{bmatrix} \vec{x}^T \\ \vec{w}^T \end{bmatrix} \vec{x} = \begin{pmatrix} \vec{x}^T\vec{x} \\ \vec{w}^T\vec{x} \end{pmatrix}$

$$= \begin{bmatrix} x_1 & x_2 \\ w_1 & w_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1x_1 + x_2x_2 \\ w_1x_1 + w_2x_2 \end{pmatrix}$$

Functions



Function: $f: \mathcal{X} \Rightarrow \mathcal{Y}$

Domain: $\mathcal{X}$ set of all possible inputs to f

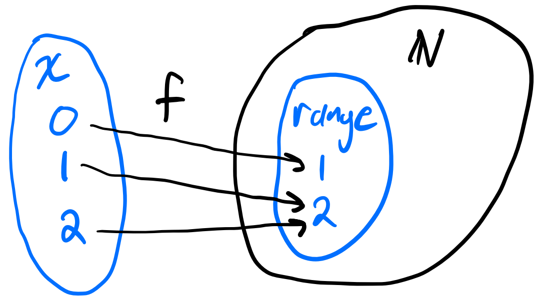
Codomain: $\mathcal{Y}$ set of all possible outputs from f

Range: set of all actual outputs from f

Ex: $f: \mathcal{X} \Rightarrow \mathcal{Y}$, $\mathcal{X} = \{0, 1, 2\}$, $\mathcal{Y} = \mathbb{N}$,

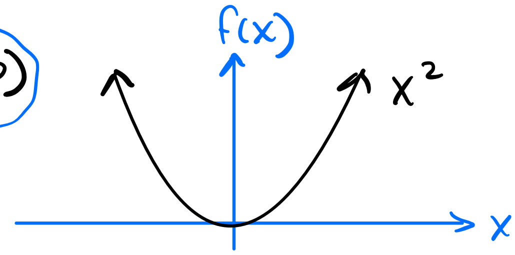
$\text{range} = \{1, 2\}$

$f(0) = 1$, $f(1) = 2$, $f(2) = 2$



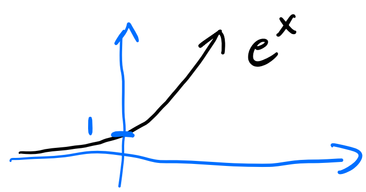
Ex: $f: \mathbb{R} \Rightarrow \mathbb{R}$, $\text{range} = \{y \in \mathbb{R} \mid y \geq 0\} = [0, \infty)$

$f(x) = x^2$ where $x \in \mathbb{R}$



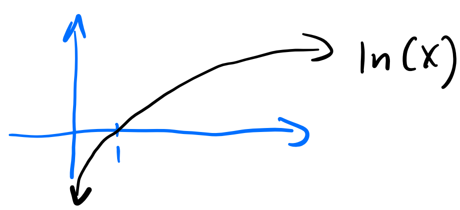
Ex: $f: \mathbb{R} \Rightarrow (0, \infty)$ $\text{range} = \{y \in \mathbb{R} \mid y > 0\} = (0, \infty)$

$f(x) = e^x = \exp(x)$



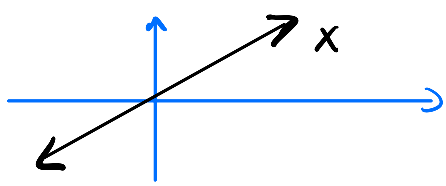
Ex: $f: \mathcal{X} \Rightarrow \mathbb{R}$, $\mathcal{X} = (0, \infty)$, $\text{range} = \mathbb{R}$

$f(x) = \ln(x) = \log_e(x)$



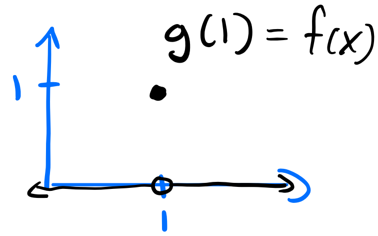
Ex: $f: \mathbb{R} \Rightarrow \mathbb{R}$, $\text{range} = \mathbb{R}$

$f(x) = \ln(e^x) = x$



Ex: $g: \mathbb{R} \to \{f \mid f: \mathcal{X} \to \mathcal{Y}\}$ $\quad \mathcal{X} = \mathbb{R},\ \mathcal{Y} = \mathbb{R}$

$$g(z) = f \quad \text{where} \quad f(x) = \begin{cases} 1 & \text{if } x = z \\ 0 & \text{otherwise} \end{cases}$$

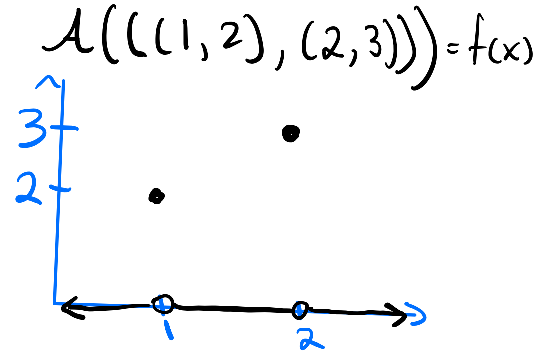


Ex: $\mathcal{A}: (\mathcal{X} \times \mathcal{Y})^n \to \{f \mid f: \mathcal{X} \to \mathcal{Y}\}$ $\quad \mathcal{X} = \mathbb{R},\ \mathcal{Y} = \mathbb{R}$

$$\mathcal{A}(D) = f, \quad D = ((\vec{x}_1, y_1), \ldots, (\vec{x}_n, y_n))$$

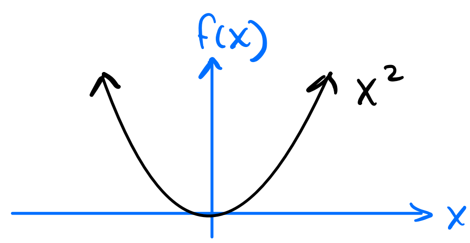
$$f(x) = \begin{cases} y_i & \text{if } x = x_i \text{ for some } i \in \{1, \ldots, n\} \\ 0 & \text{otherwise} \end{cases}$$

$\overbrace{\mathcal{A}(D)}^{=f}(x)$

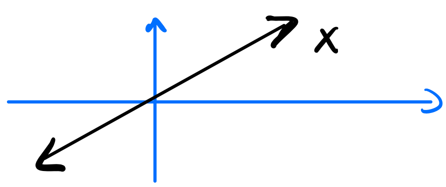


Continuous function: a function without abrupt changes

Ex: $f(x) = x^2$

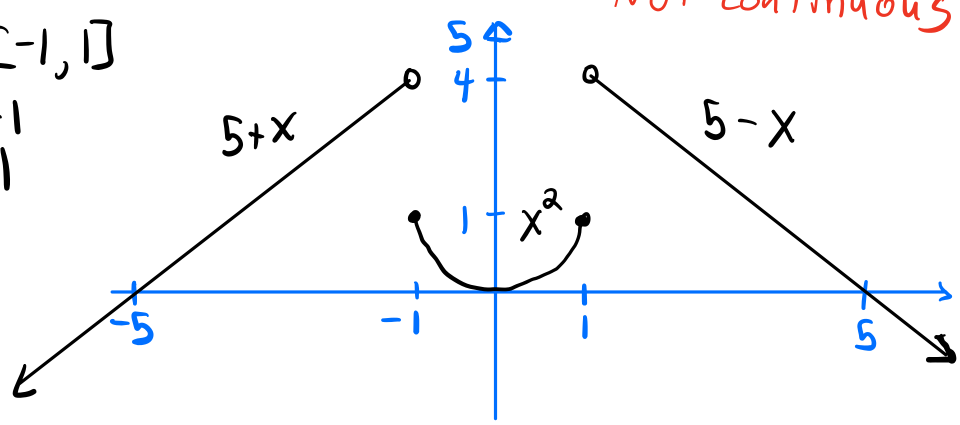


$f(x) = x$



$$f(x) = \begin{cases} x^2 & x \in [-1, 1] \\ 5 + x & x < -1 \\ 5 - x & x > 1 \end{cases}$$

Not continuous



Summation and Integration: accumulation of values in a set

Motivation: Needed to define expected value

Summation: Σ over discrete sets

Ex: $\mathcal{X}=\{0,1,2\}$, $\sum_{x\in\mathcal{X}} x = 0+1+2=3$

$f(x)=x^2$, $\sum_{x\in\mathcal{X}} f(x) = 0^2+1^2+2^2=5$

$\mathcal{X}=(x_1,x_2,\ldots,x_n)$, $\sum_{i=1}^{n} x_i = \sum_{x\in\mathcal{X}} x = x_1+x_2+\cdots+x_n$

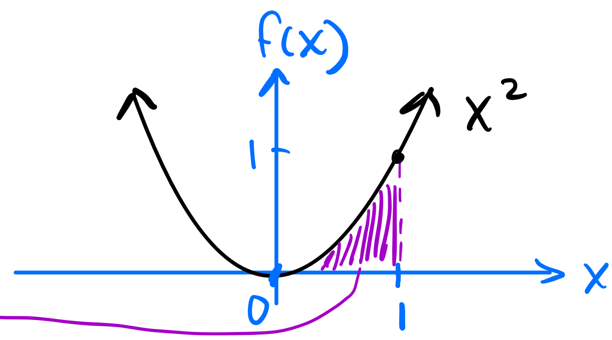
Integration: $\int$ over continuous sets

Ex: $\mathcal{X}=[a,b]$, $f:\mathcal{X}\to\mathbb{R}$

$$\int_{\mathcal{X}} f(x)\,dx = \int_a^b f(x)\,dx$$

if $a=0, b=1, f(x)=x^2$, then

$$\int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

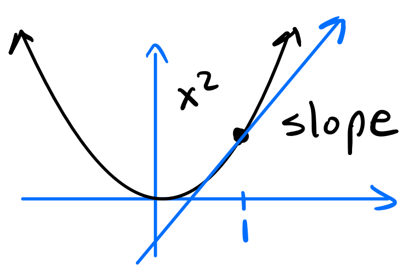


Derivatives: rate of change of a function

written f' or $\frac{df}{dx}$ for a function f

Motivation: We want our learner to pick the best predictor (optimization)

Ex: $f(x)=x^2$, $f'(x)=2x=\frac{df}{dx}(x)$



$f(x) = x^a,\ f'(x) = ax$

$f(x) = e^x,\ f'(x) = e^x$

$f(x) = \ln(x),\ f'(x) = \frac{1}{x}$

Chain rule $f(x) = g(h(x)),\ f'(x) = g'(h(x))\, h'(x)$

Ex: $f(x) = \exp(x^2),\ g(h(x)) = \exp(h(x)),\ h(x) = x^2$

$g'(h(x)) = \exp(h(x)),\ h'(x) = 2x$

$f'(x) = \exp(x^2)\, 2x$

Partial Derivative: Derivative of a function that takes as input more than one variable

$\frac{\partial f}{\partial x_1}(x_1)$ is the partial derivative of $f(x_1, x_2)$ with respect to x_1

Ex: $f(x_1, x_2) = 2x_1 + 3x_2^2,\ \frac{\partial f}{\partial x_1}(x_1) = 2,\ \frac{\partial f}{\partial x_2}(x_2) = 6x_2$

Ex: $f(\vec{x}) = f(x_1, \dots, x_d) = x_1 w_1 + \dots + x_d w_d = \sum_{i=1}^{d} x_i w_i = \vec{x}^T \vec{w}$

where $\vec{x} = (x_1, \dots, x_d)^T \in \mathbb{R}^d$ are the variables

and $\vec{w} = (w_1, \dots, w_d)^T \in \mathbb{R}^d$ are some fixed numbers (constants)

$\frac{\partial f}{\partial x_1}(x_1) = w_1, \dots, \frac{\partial f}{\partial x_d}(x_d) = w_d$

Gradient: A vector of all the partial derivatives

$$\nabla f(x_1, x_2) = \left(\frac{\partial f}{\partial x_1}(x_1), \frac{\partial f}{\partial x_2}(x_2) \right)^T$$

<u>Ex</u>: $f(\vec{x}) = f(x_1, \ldots, x_d) = x_1 w_1 + \ldots + x_d w_d = \vec{x}^T \vec{w}$

$$\nabla f(\vec{x}) = \left(\frac{\partial f}{\partial x_1}(x_1), \ldots, \frac{\partial f}{\partial x_d}(x_d) \right)^T$$

$$= (w_1, \ldots, w_d)^T$$