Review for Quiz

Chapter 2 (Probability)
Chapter 3 (Estimation):
Bias, Variance, Concentration Inequalities

CMPUT 267: Basics of Machine Learning

Logistics

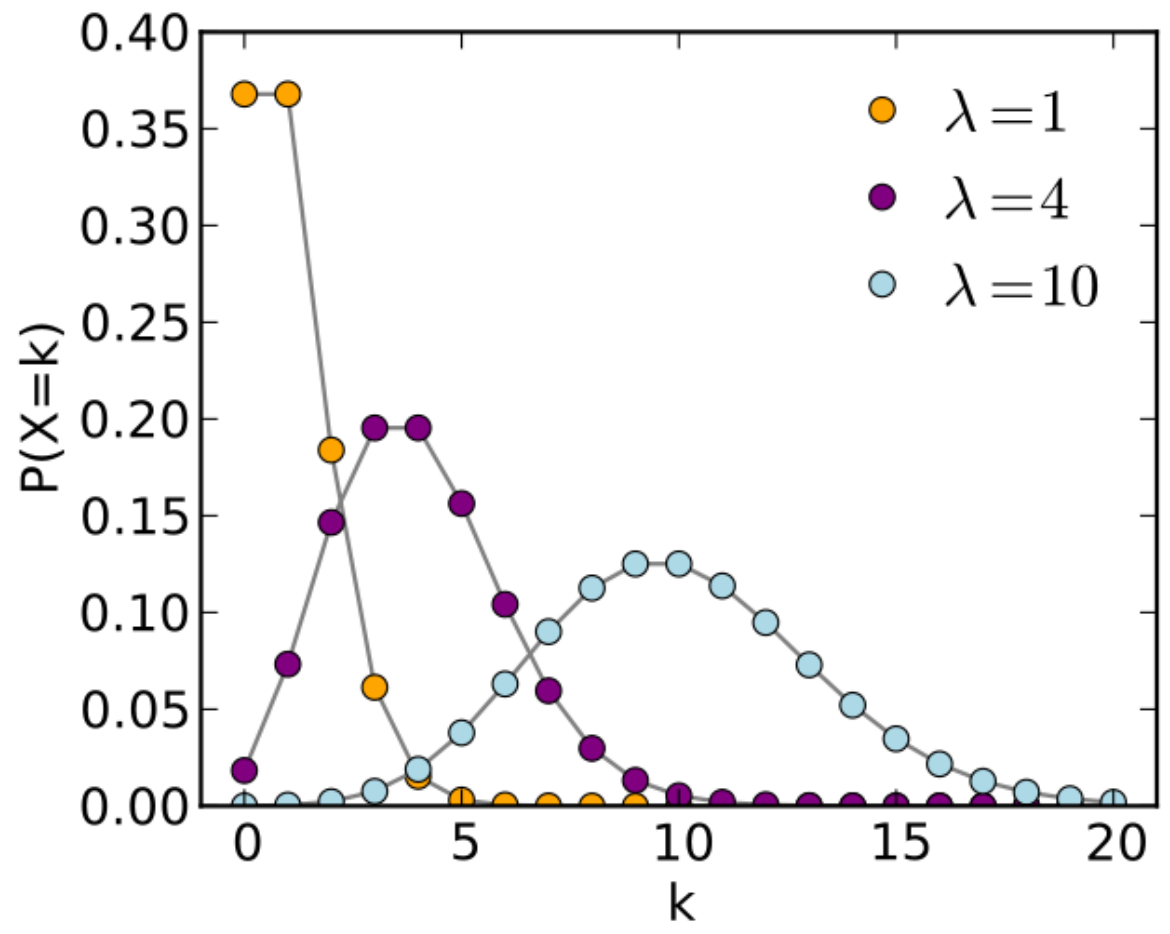
- Quiz during class on Thursday
  - Join 10 minutes early on Zoom lecture
  - Or come to class physically
- Any questions/issues with Assignment 2?

Language of Probabilities

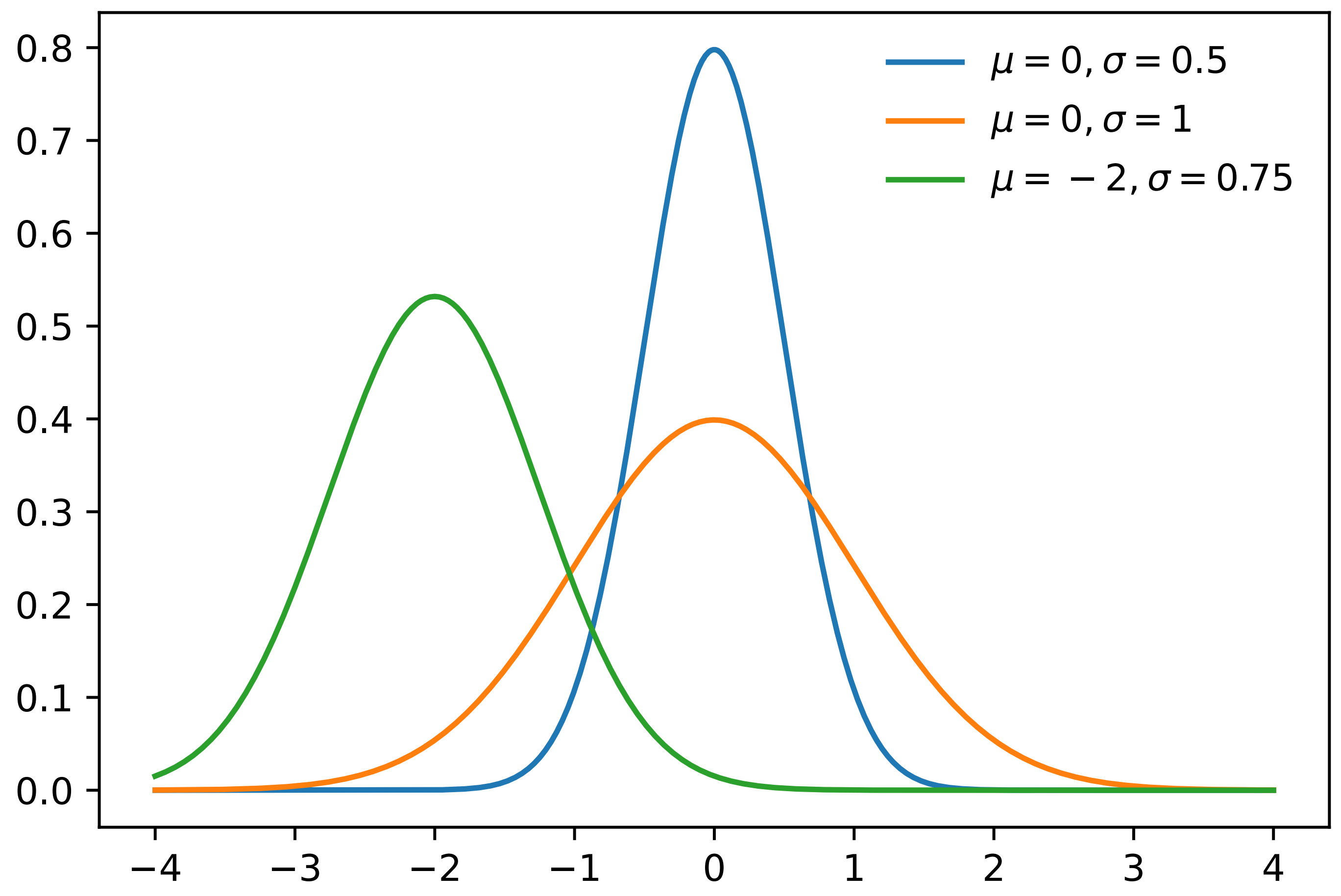
- Define random variables, and their distributions
  - Then can formally reason about them
- Express our beliefs about behaviour of these RVs, and relationships to other RVs
- Examples:
  - p(x) Gaussian means we believe X is Gaussian distributed
  - p(y | X = x)—or written p(y | x)— is Gaussian this means that conditioned on x, y is Gaussian; but p(y) might not be Gaussian
  - p(w) and p(w | Data)

PMFs and PDFs

- Discrete RVs have PMFs
  - outcome space: e.g, $\Omega = \{1,2,3,4,5,6\}$
  - examples pmfs: probability tables, Poisson $p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$



- Continuous RVs have PDFs
  - outcome space: e.g., $\Omega = [0,1]$
  - example pdf: Gaussian, Gamma



A few questions

- Do PMFs p(x) have to output values between [0,1]?
- Do PDFs p(x) have to output values between [0,1]?
- What other condition(s) are put on a function p to make it a valid pmf or pdf?
- Is the following function a pdf or a pmf?
- $p(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b, \\ 0 & \text{otherwise.} \end{cases}$ i.e., $p(x) = \frac{1}{b-a}$ for $x \in [a, b]$

How would you define a uniform distribution for a discrete RV

- Imagine $x \in \{1,2,3,4,5\}$
- What is the uniform pmf for this outcome space?
- $$p(x) = \begin{cases} \frac{1}{5} & \text{if } x \in \{1,2,3,4,5\}, \\ 0 & \text{otherwise.} \end{cases}$$

How do you answer this probabilistic question?

- For continuous RV X with a uniform distribution and outcome space [0,10], what is the probability that X is greater than 7?

$$\Pr(X > 7) = \int_7^{10} p(x)dx = \int_7^{10} \frac{1}{10}dx$$

$$= \frac{1}{10}\int_7^{10} dx = \frac{1}{10}x\,|_7^{10}$$

$$= \frac{3}{10}$$

-

Multivariate Setting

- Conditional distribution, $p(y \mid x) = \dfrac{p(x, y)}{p(x)}$, Marginal $p(y) = \sum_{x \in \mathcal{X}} p(x, y)$
- Chain Rule $p(x, y) = p(y \mid x)p(x) = p(x \mid y)p(y)$
- Bayes Rule $p(y \mid x) = \dfrac{p(x \mid y)p(y)}{p(x)}$
- Law of total probability $p(y) = \sum_{x \in \mathcal{X}} p(y|x)p(x)$
- **Question**: How do you get the law of total probability from the chain rule?

Expectations

$$\mathbb{E}[f(X)] = \begin{cases} \sum_{x \in \mathscr{X}} f(x)p(x) & \text{if } X \text{ is discrete,} \\ \int_{\mathscr{X}} f(x)p(x)\, dy & \text{if } X \text{ is continuous.} \end{cases}$$

Eg: $\mathscr{X} = \{1,2,3,4,5\}$, $f(x) = x^2$, $Y = f(X)$, map $\{1,2,3,4,5\} \to \{1,4,9,16,25\}$,
$p(y)$ determined by $p(x)$, e.g, $p(Y=4) = p(X=2)$

Eg: $\mathscr{X} = \{-1,0,1\}$, $f(x) = |x|$, $Y = f(X)$, map $\{-1,0,1\} \to \{0,1\}$
$p(Y=1) = p(X=-1) + p(X=1)$, $\mathbb{E}[Y] = \sum_{y \in 0,1} y p(y) = \sum_{x \in \{-1,0,1\}} f(x)p(x)$

Conditional Expectations

Definition:

The **expected value of Y conditional on $X = x$** is

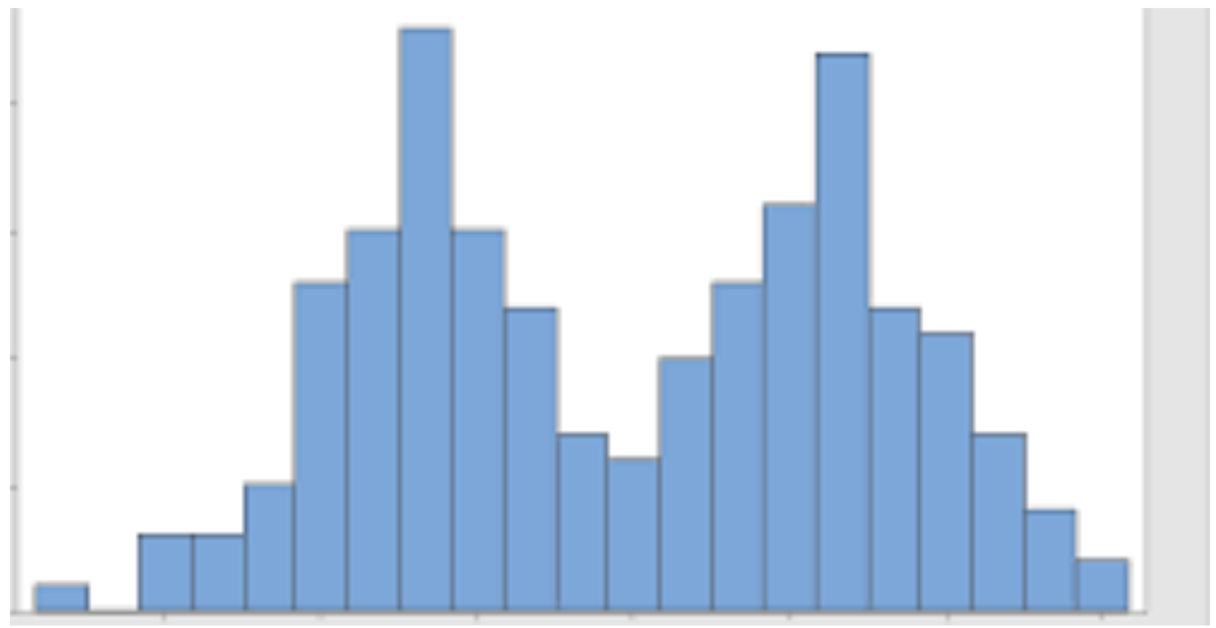
$$\mathbb{E}[Y \mid X = x] = \begin{cases} \sum_{y \in \mathcal{Y}} y p(y \mid x) & \text{if } Y \text{ is discrete,} \\ \int_{\mathcal{Y}} y p(y \mid x)\, dy & \text{if } Y \text{ is continuous.} \end{cases}$$

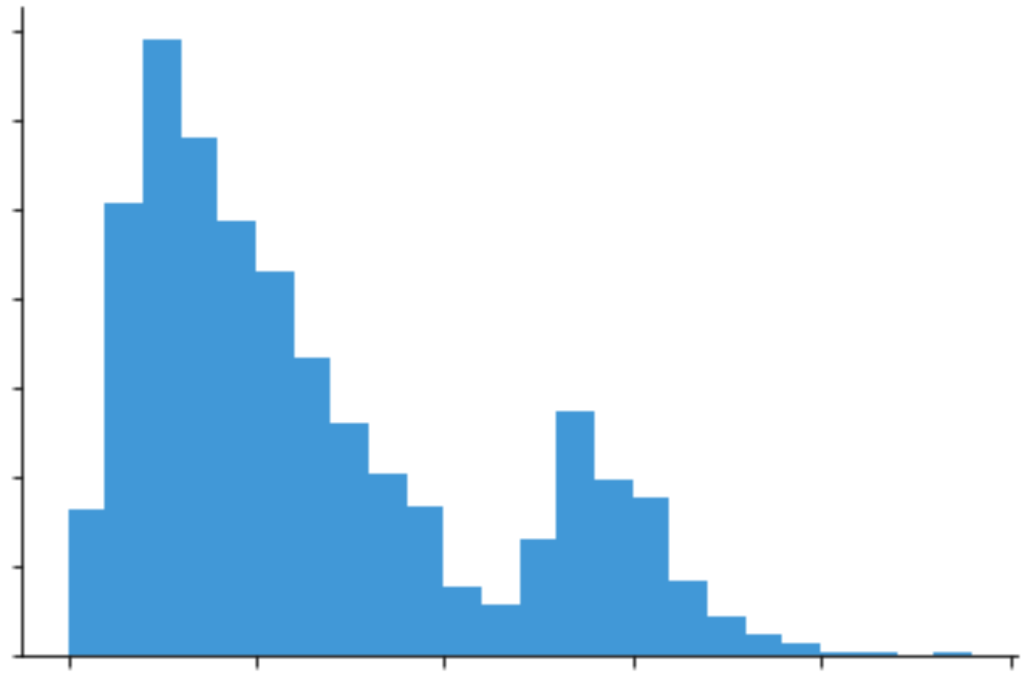
Conditional Expectation Example

- X is the type of a book, 0 for fiction and 1 for non-fiction
  - $p(X=1)$ is the proportion of all books that are non-fiction
- Y is the number of pages
  - $p(Y=100)$ is the proportion of all books with 100 pages
- $p(y \mid X=0)$ is different from $p(y \mid X=1)$
- $\mathbb{E}[Y \mid X=0]$ is different from $\mathbb{E}[Y \mid X=1]$
  - e.g. $\mathbb{E}[Y \mid X=0]=70$ is different from $\mathbb{E}[Y \mid X=1]=150$

Conditional Expectation Example (cont)

- $p(y|X=0)$ $p(y|X=1)$





- $\mathbb{E}[Y|X=0]$ is the expectation over Y under distribution $p(y|X=0)$
- $\mathbb{E}[Y|X=1]$ is the expectation over Y under distribution $p(y|X=1)$

What if Y is dollars earned?

- Y is now a continuous RV
- What is $p(y \mid x)$?

What if Y is dollars earned?

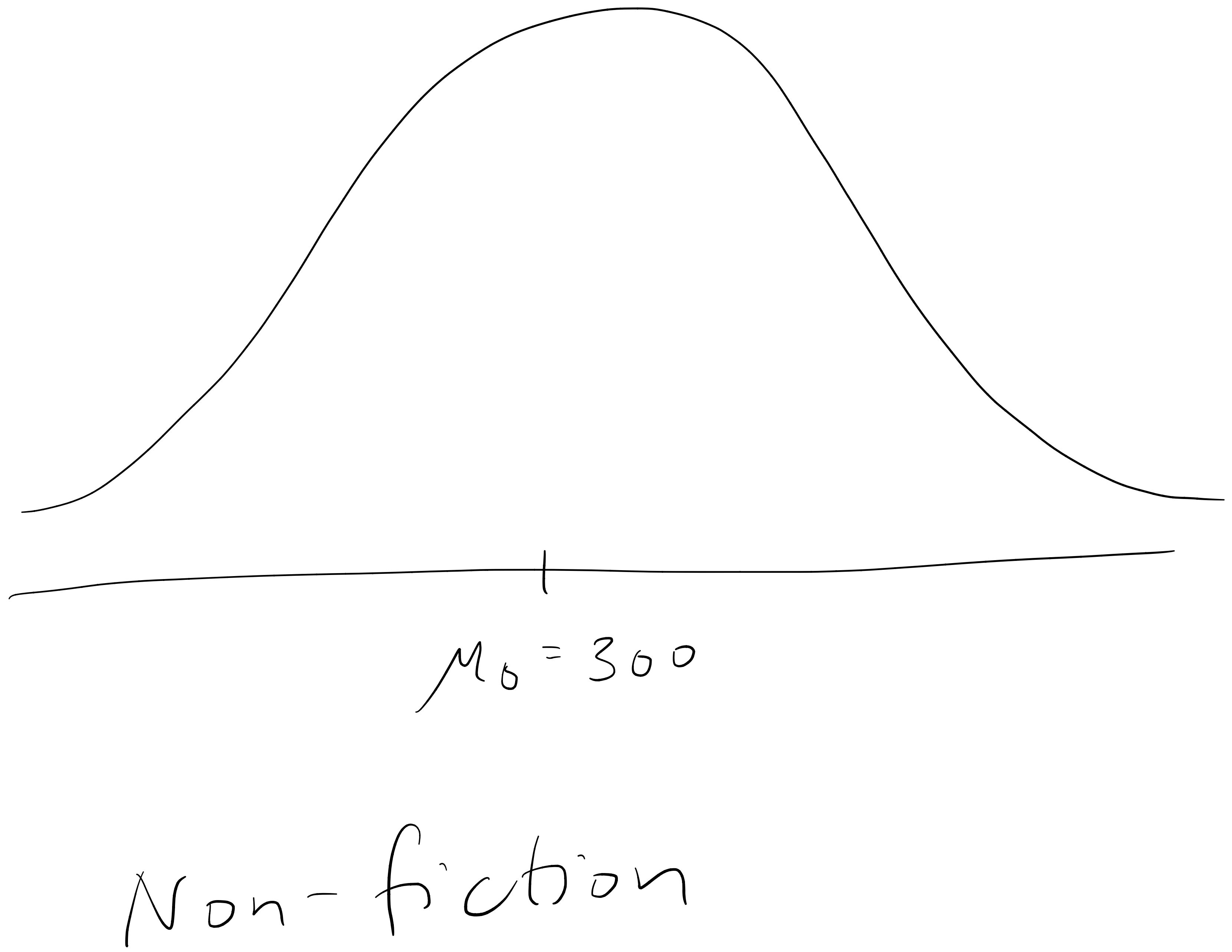
- Y is now a continuous RV
- Notice that $p(y \mid x)$ is defined by $p(y \mid X = 0)$ and $p(y \mid X = 1)$
- What might be a reasonable choice for $p(y \mid X = 0)$ and $p(y \mid X = 1)$?

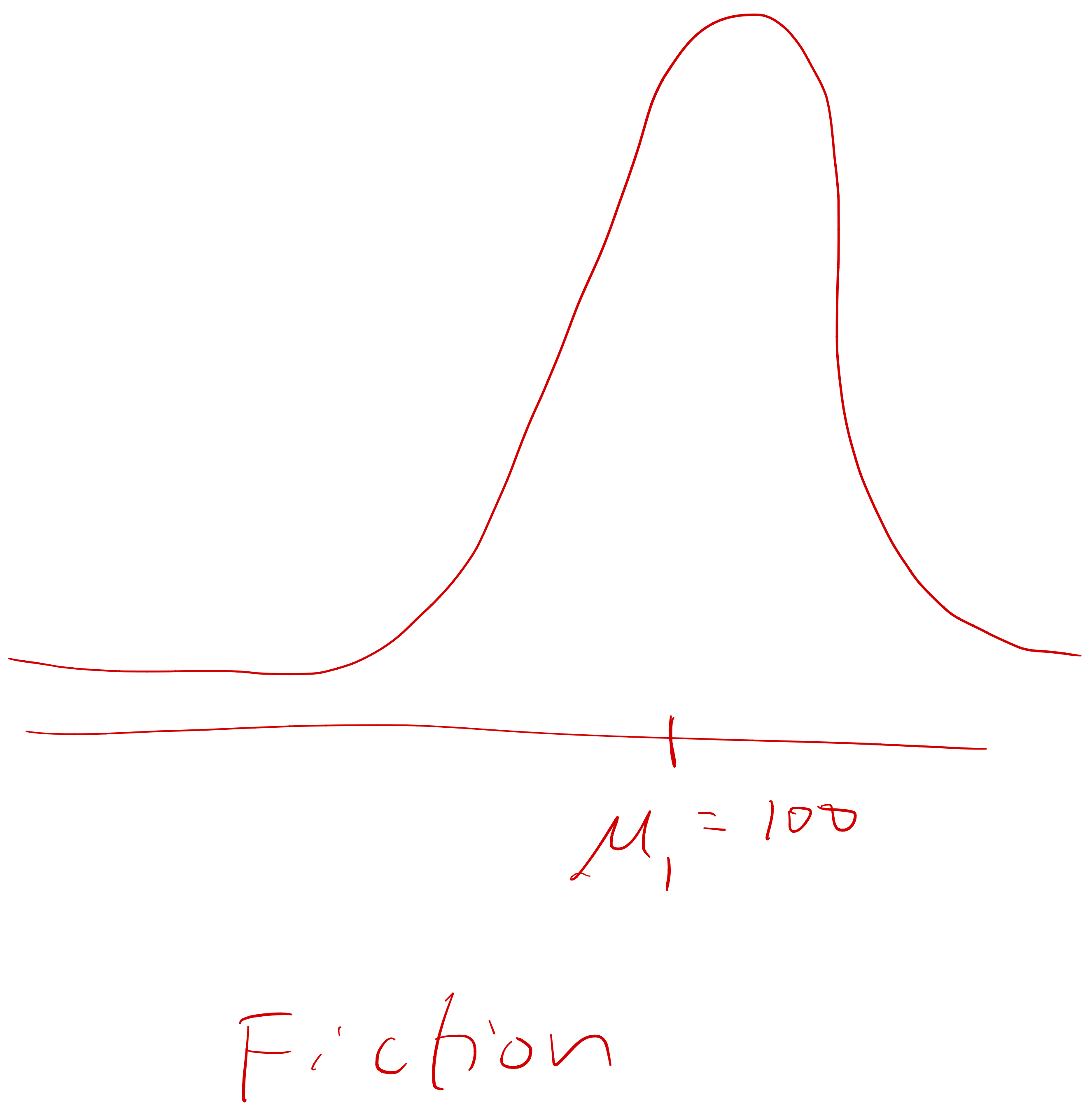
What if Y is dollars earned?

- Notice that $p(y \mid x)$ is defined by $p(y \mid X = 0)$ and $p(y \mid X = 1)$

$p(y \mid X=0) = \mathcal{N}(\mu_0, \sigma_0^2)$

$p(y \mid X=1) = \mathcal{N}(\mu_1, \sigma_1^2)$





Exercise

- Come up with an example of X and Y, and give possible choice for $p(y \mid x)$
- Do you need to know $p(x)$ to specify $p(y \mid x)$?

Properties of Expectations

- Linearity of expectation:
  - $\mathbb{E}[cX] = c\mathbb{E}[X]$ for all constant c
  - $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
- Products of expectations of **independent** random variables X, Y:
  - $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Law of Total Expectation:
  - $\mathbb{E}\left[\mathbb{E}\left[Y \mid X\right]\right] = \mathbb{E}[Y]$

Properties of Expectations for X and Y independent

$$
\begin{aligned}
\mathbb{E}[XY] &= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) xy \\
&= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(y | x) p(x) xy \\
&= \sum_{x \in \mathcal{X}} xp(x) \sum_{y \in \mathcal{Y}} p(y | x) y \\
&= \sum_{x \in \mathcal{X}} xp(x) \mathbb{E}[Y | x] \\
&= \sum_{x \in \mathcal{X}} xp(x) \mathbb{E}[Y] \qquad \text{since X and Y independent} \\
&= \mathbb{E}[X] \mathbb{E}[Y]
\end{aligned}
$$

Variance

Definition: The **variance** of a random variable is

$$\text{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right].$$

i.e., $\mathbb{E}[f(X)]$ where $f(x) = (x - \mathbb{E}[X])^2$.

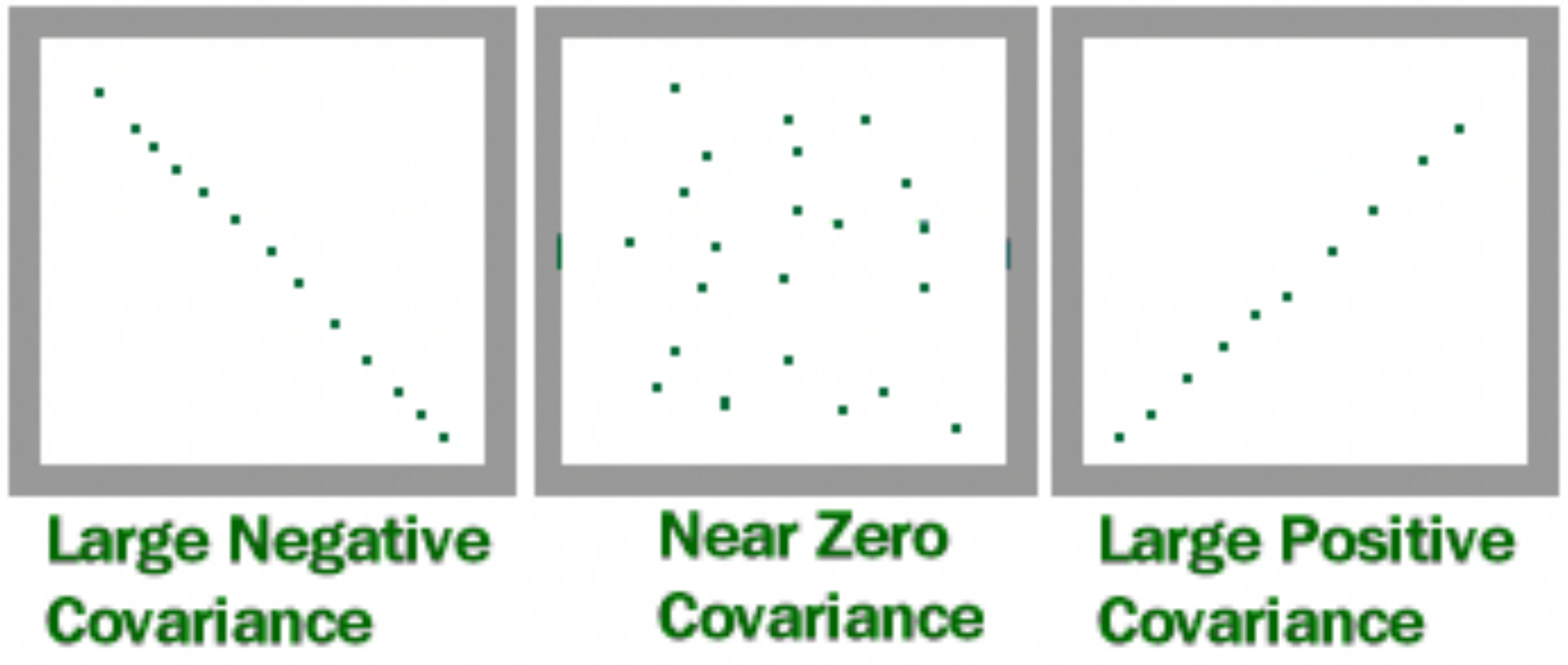
Equivalently,

$$\text{Var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2$$

Covariance

Definition: The covariance of two random variables is

$$\text{Cov}(X, Y) = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right]$$
$$= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\,.$$



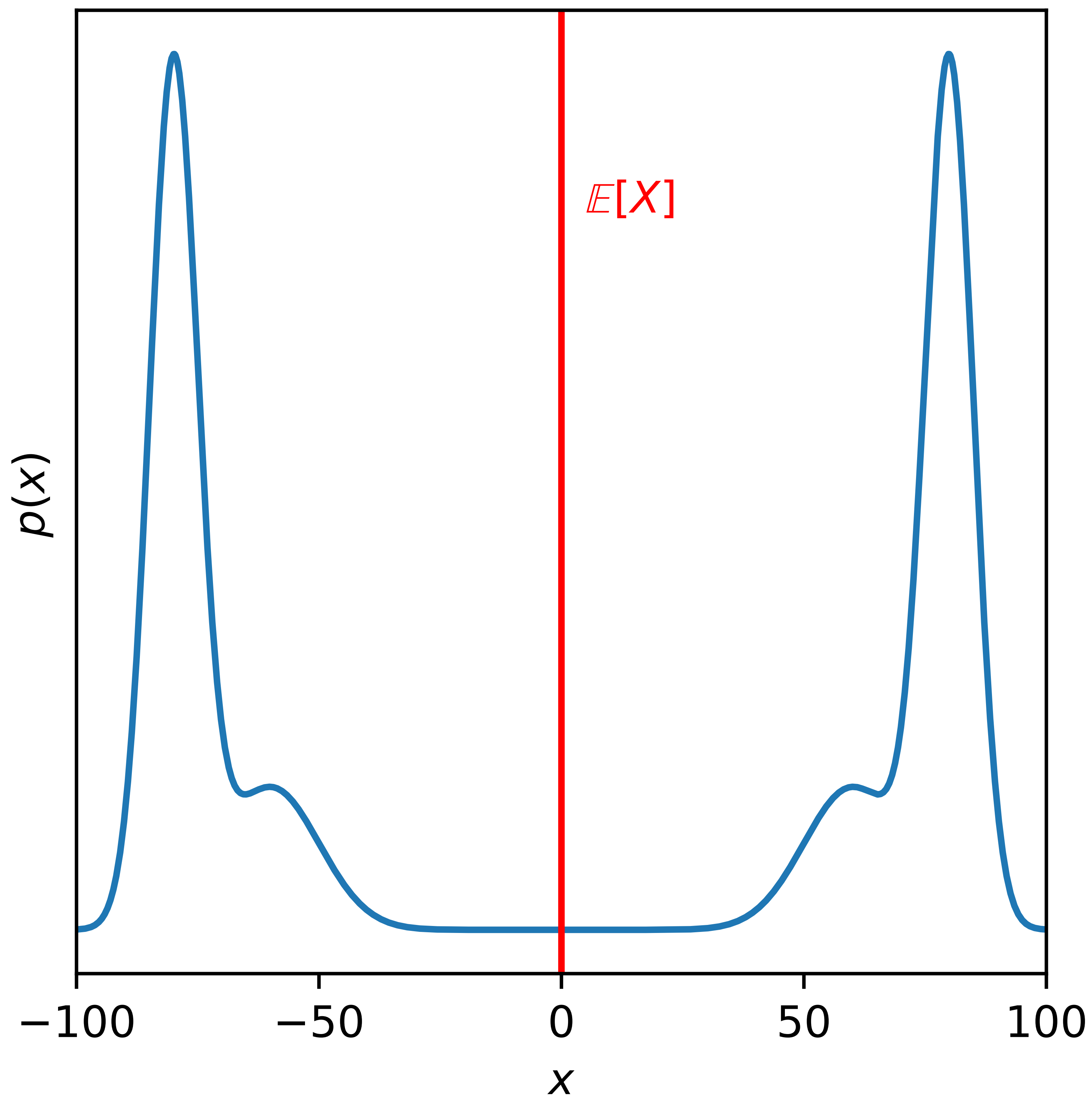
Properties of Variances

- Var$[c] = 0$ for constant c
- Var$[cX] = c^2$Var$[X]$ for constant c
- Var$[X + Y] =$ Var$[X] +$ Var$[Y] + 2$Cov$[X, Y]$
- For **independent** X, Y, because Cov$[X, Y] = 0$
  Var$[X + Y] =$ Var$[X] +$ Var$[Y]$

Estimators

Definition: An **estimator** is a procedure for estimating an unobserved quantity based on data.

Example: Estimating $\mathbb{E}[X]$ for r.v. $X \in \mathbb{R}$.



random variable!

Questions:

Suppose we can observe a different variable Y. Is Y a good estimator of $\mathbb{E}[X]$ in the following cases? Why or why not?

1. $Y \sim$ Uniform$[0,10]$
2. $Y = \mathbb{E}[X] + Z,$ where $Z \sim N(0,100^2)$
3. $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$, for $X_i \sim p$

Independent and Identically Distributed (i.i.d.) Samples

- We usually won't try to estimate anything about a distribution based on only a single sample
- Usually, we use **multiple samples** from the **same distribution**
  - *Multiple samples:* This gives us more information
  - *Same distribution:* We want to learn about a single population
- One additional condition: the samples must be **independent**

Definition: When a set of random variables are $X_1, X_2, \ldots$ are all independent, and each has the same distribution $X \sim F$, we say they are **i.i.d.** (independent and identically distributed), written

$$X_1, X_2, \ldots \overset{i.i.d.}{\sim} F.$$

Estimating Expected Value via the Sample Mean

Example: We have n i.i.d. samples from the same distribution F,

$$X_1, X_2, \ldots, X_n \overset{i.i.d}{\sim} F,$$

with $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$ for each X_i.

We want to estimate μ.

Let's use the **sample mean** $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ to estimate μ.

$$\begin{aligned}\mathbb{E}[\bar{X}] &= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \\ &= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_i] \\ &= \frac{1}{n}\sum_{i=1}^{n} \mu \\ &= \frac{1}{n} n\mu \\ &= \mu . \qquad \blacksquare\end{aligned}$$

Bias

Definition: The bias of an estimator $\hat{X}$ is its expected difference from the true value of the estimated quantity X:

$$\text{Bias}(\hat{X}) = \mathbb{E}[\hat{X}] - \mathbb{E}[X]$$

- Bias can be positive or negative or zero
- When $\text{Bias}(\hat{X}) = 0$, we say that the estimator $\hat{X}$ is **unbiased**

Questions:

What is the **bias** of the following estimators of $\mathbb{E}[X]$?

1. $Y \sim \text{Uniform}[0,10]$
2. $Y = \mathbb{E}[X] + Z$, where $Z \sim \text{Uniform}[0,1]$
3. $Y = \mathbb{E}[X] + Z$, where $Z \sim N(0,100^2)$
4. $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$

Variance of the Estimator

- Intuitively, more samples should make the estimator "closer" to the estimated quantity
- We can formalize this intuition partly by characterizing the **variance $\text{Var}[\hat{X}]$ of the estimator itself**.
  - The variance of the estimator should decrease as the number of samples increases
- **Example:** $\bar{X}$ for estimating μ:
  - The variance of the estimator shrinks linearly as the number of samples grows.

$$
\begin{aligned}
\text{Var}[\bar{X}] &= \text{Var}\left[\frac{1}{n}\sum_{i=1}^{n} Xi\right] \\
&= \frac{1}{n^2}\text{Var}\left[\sum_{i=1}^{n} X_i\right] \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}[X_i] \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 \\
&= \frac{1}{n^2}n\sigma^2 \quad = \frac{1}{n}\sigma^2.
\end{aligned}
$$

Mean-Squared Error

- **Bias:** whether an estimator is correct **in expectation**
- **Consistency:** whether an estimator is correct **in the limit of infinite data**
- **Convergence rate:** how fast the estimator **approaches its own mean**
  - For an **unbiased** estimator, this is also how fast its **error bounds** shrink
- We don't necessarily care about an estimator's being unbiased.
  - Often, what we care about is our estimator's **accuracy in expectation**

Definition: Mean squared error of an estimator $\hat{X}$ of a quantity X:

$$\text{MSE}(\hat{X}) = \mathbb{E}\left[(\hat{X} - \mathbb{E}[X])^2\right]$$

different!

Bias-Variance Tradeoff

$$\text{MSE}(\hat{X}) = \text{Var}[\hat{X}] + \text{Bias}(\hat{X})^2$$

- If we can decrease bias without increasing variance, error goes down
- If we can decrease variance without increasing bias, error goes down
- **Question:** Would we ever want to **increase bias**?
- *YES.* If we can increase (squared) bias in a way that **decreases variance more**, then error goes down!
  - **Interpretation:** Biasing the estimator toward values that are **more likely to be true** (based on **prior information)**

Downward-biased Mean Estimation

Example: Let's estimate μ given i.i.d $X_1, \ldots, X_n$ with $\mathbb{E}[X_i] = \mu$ using: $Y = \frac{1}{n+100} \sum_{i=1}^{n} X_i$

This estimator is **biased**:

$$\mathbb{E}[Y] = \mathbb{E}\left[\frac{1}{n+100}\sum_{i=1}^{n} X_i\right]$$

$$= \frac{1}{n+100}\sum_{i=1}^{n}\mathbb{E}[X_i]$$

$$= \frac{n}{n+100}\mu$$

$$\text{Bias}(Y) = \frac{n}{n+100}\mu - \mu = \frac{-100}{n+100}\mu$$

This estimator has **low variance**:

$$\text{Var}(Y) = \text{Var}\left[\frac{1}{n+100}\sum_{i=1}^{n} X_i\right]$$

$$= \frac{1}{(n+100)^2}\text{Var}\left[\sum_{i=1}^{n} X_i\right]$$

$$= \frac{1}{(n+100)^2}\sum_{i=1}^{n}\text{Var}[X_i]$$

$$= \frac{n}{(n+100)^2}\sigma^2$$

Estimating μ Near 0

Example: Suppose that $\sigma = 1$, $n = 10$, and $\mu = 0.1$

$\text{Bias}(\bar{X}) = 0$

$$\text{MSE}(\bar{X}) = \text{Var}(\bar{X}) + \text{Bias}(\bar{X})^2$$

$$= \text{Var}(\bar{X}) \qquad \text{Var}(\bar{X}) = \frac{\sigma^2}{n}$$

$$= \frac{1}{10}$$

$$\text{MSE}(Y) = \text{Var}(Y) + \text{Bias}(Y)^2$$

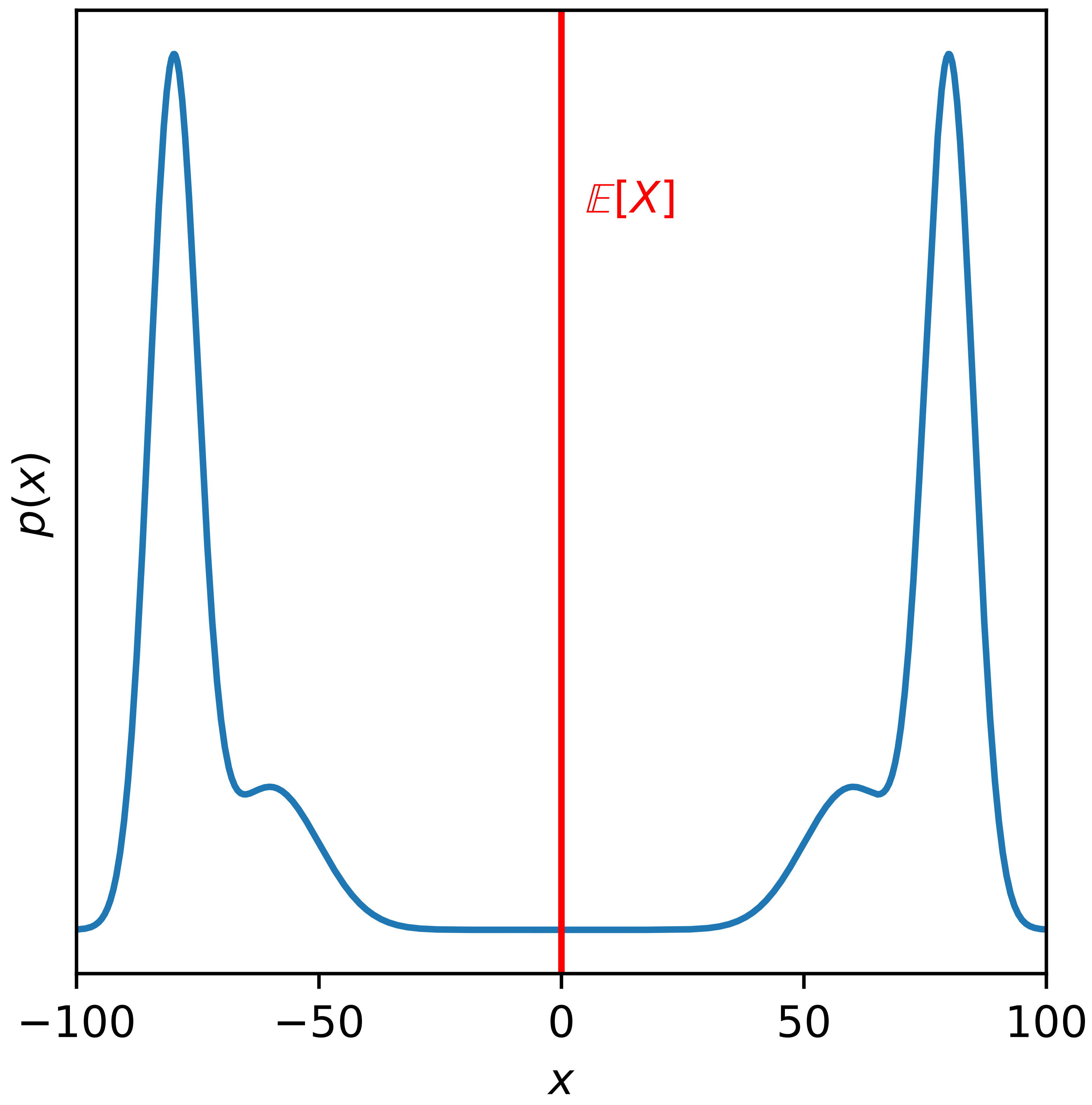
$$= \frac{n}{(n+100)^2}\sigma^2 + \left(\frac{100}{n+100}\mu\right)^2$$

$$= \frac{10}{110^2} + \left(\frac{100}{110}0.1\right)^2$$

$$\approx 9 \times 10^{-4}$$

Exercise: What is the variance of these estimators?

Example: Estimating $\mathbb{E}[X]$ for r.v. $X \in \mathbb{R}$.



Questions:

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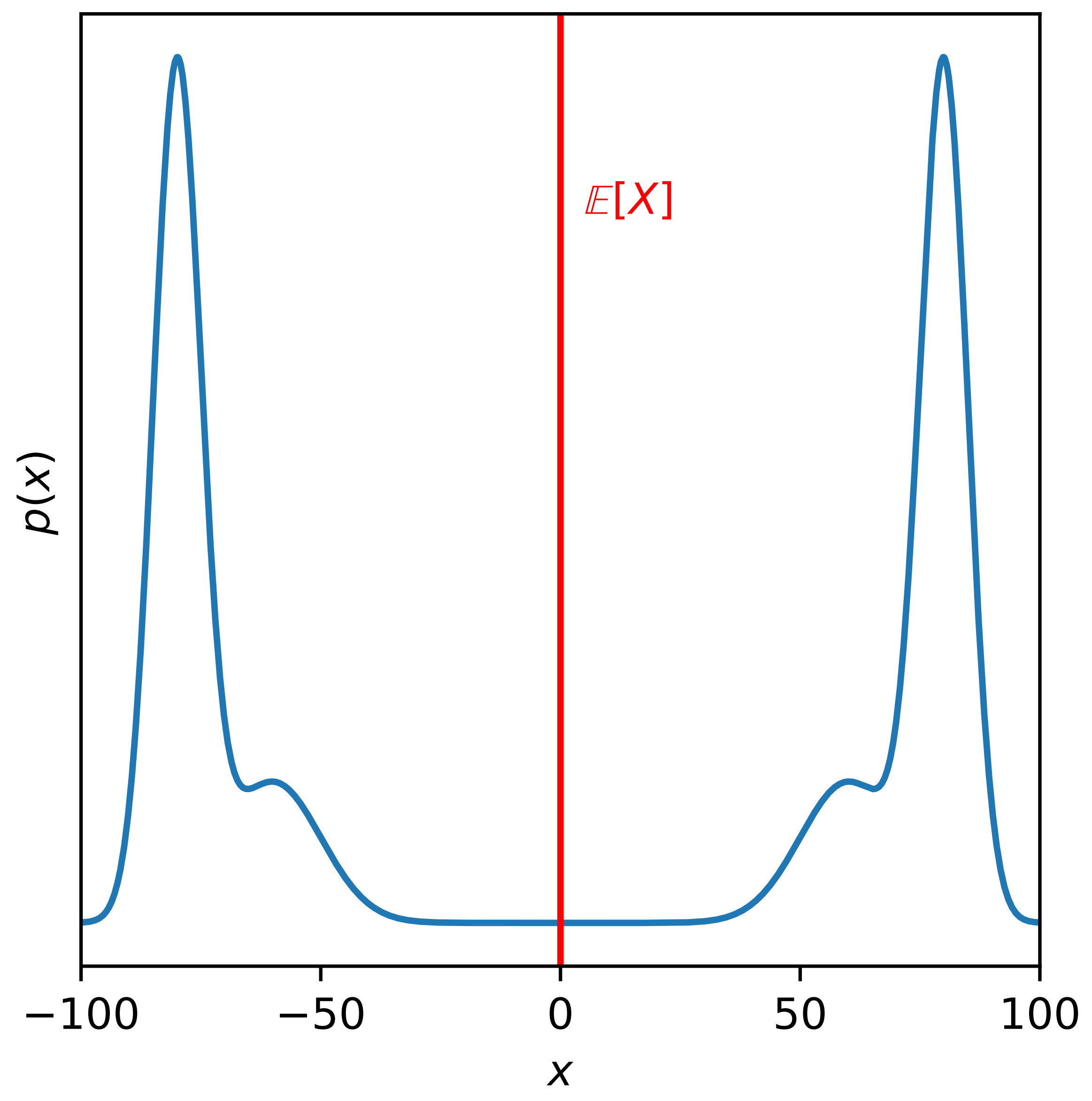
2. $Y = \mathbb{E}[X] + Z,$ where $Z \sim N(0,100^2)$
3. $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$, for $X_i \sim p$

Exercise: What is the variance of these estimators?



$$\mathrm{Var}\left[\frac{1}{n}\sum_{i=1}^{n} Xi\right] = \frac{1}{n}\sigma^2.$$

Estimators:

1. $Y_1 \sim \text{Uniform}[0,10]$
2. $Y_2 = \mathbb{E}[X] + Z,$ where $Z \sim N(0,100^2)$
3. $Y_3 = \frac{1}{n}\sum_{i=1}^{n} X_i,$ for $X_i \sim p$

$$\text{Var}(Y_1) = \frac{1}{12}(10-0)^2 = \frac{100}{12} = 8.\bar{3}$$

$$\text{Var}(Y_2) = \text{Var}(\mathbb{E}[X] + Z) = ?$$

$$\text{Var}(Y_3) = \frac{\sigma^2}{n}$$

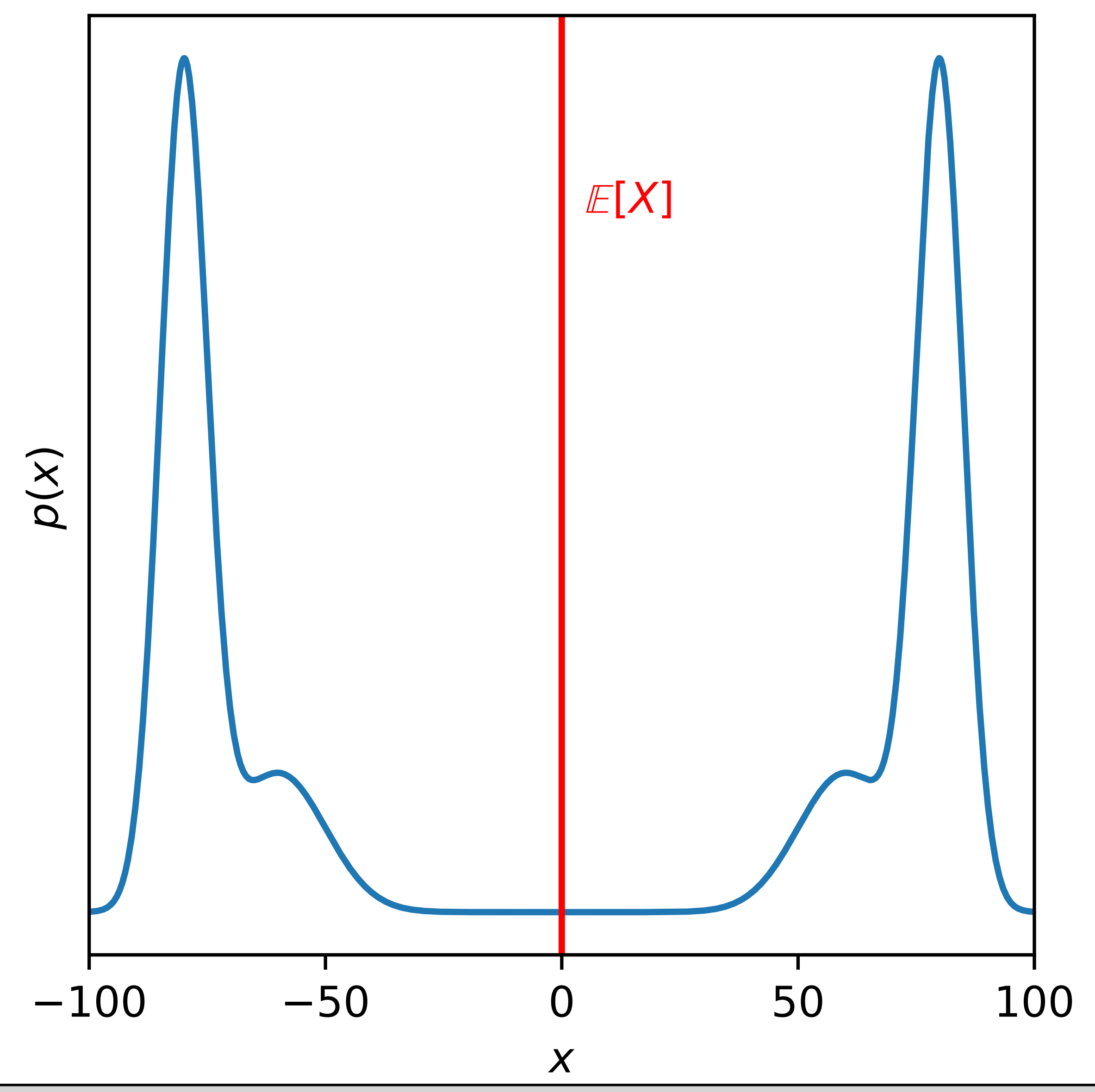
Exercise: What is the variance of these estimators?

Estimators:

1. $Y_1 \sim$ Uniform$[0,10]$
2. $Y_2 = \mathbb{E}[X] + Z,$ where $Z \sim N(0,100^2)$
3. $Y_3 = \frac{1}{n}\sum_{i=1}^{n} X_i$, for $X_i \sim p$

$$\begin{aligned}\text{Var}(Y_2) &= \text{Var}(\mathbb{E}[X] + Z) \\ &= \text{Var}(Z) && \triangleright\ \text{Var}(c + Y) = \text{Var}(Y) \\ &= 100^2\end{aligned}$$

MSE of these estimators



$$\text{Var}(Y_1) = \frac{1}{12}(10-0)^2 = \frac{100}{12} = 8.\bar{3}$$

$$\text{Bias}(Y_1) = \mathbb{E}[Y_1] - \mathbb{E}[X] = 5$$

$$\text{Var}(Y_2) = \text{Var}(\mathbb{E}[X] + Z) = 100^2$$

$$\text{Bias}(Y_2) = \mathbb{E}[Y_2] - \mathbb{E}[X] = 0$$

$$\text{Var}(Y_3) = \frac{\sigma^2}{n}$$

$$\text{Bias}(Y_3) = 0$$

Estimators:

1. $Y_1 \sim$ Uniform$[0,10]$
2. $Y_2 = \mathbb{E}[X] + Z,$ where $Z \sim N(0,100^2)$
3. $Y_3 = \frac{1}{n}\sum_{i=1}^{n} X_i,$ for $X_i \sim p$

$$\text{MSE}(Y_1) = 5^2 + 8.\bar{3} = 33.\bar{3}$$

$$\text{MSE}(Y_2) = 0 + 100^2 = 10000$$

$$\text{MSE}(Y_3) = 0 + \frac{\sigma^2}{n}$$

$$\text{MSE}(\hat{X}) = \text{Var}[\hat{X}] + \text{Bias}(\hat{X})^2$$

Concentration Inequalities

- We would like to be able to claim $\Pr\left(\left|\bar{X}-\mu\right| < \epsilon\right) > 1-\delta$ for some $\delta, \epsilon > 0$
- $\text{Var}[\bar{X}] = \frac{1}{n}\sigma^2$ means that with "enough" data, $\Pr\left(\left|\bar{X}-\mu\right| < \epsilon\right) > 1-\delta$ for *any* $\delta, \epsilon > 0$ that we pick
- Suppose we have $n = 10$ samples, and we know $\sigma^2 = 81$; so $\text{Var}[\bar{X}] = 8.1$.
- **Question:** What is $\Pr\left(\left|\bar{X}-\mu\right| < 2\right)$?

Knowing the Variance Is Not Enough

Knowing $\mathrm{Var}[\bar{X}] = 8.1$ is **not enough** to compute $\Pr(|\bar{X} - \mu| < 2)$!

Examples:

$$p(\bar{x}) = \begin{cases} 0.9 & \text{if } \bar{x} = \mu \\ 0.05 & \text{if } \bar{x} = \mu \pm 9 \end{cases} \implies \mathrm{Var}[\bar{X}] = 8.1 \text{ and } \Pr(|\bar{X} - \mu| < 2) = 0.9$$

$$p(\bar{x}) = \begin{cases} 0.999 & \text{if } \bar{x} = \mu \\ 0.0005 & \text{if } \bar{x} = \mu \pm 90 \end{cases} \implies \mathrm{Var}[\bar{X}] = 8.1 \text{ and } \Pr(|\bar{X} - \mu| < 2) = 0.999$$

$$p(\bar{x}) = \begin{cases} 0.1 & \text{if } \bar{x} = \mu \\ 0.45 & \text{if } \bar{x} = \mu \pm 3 \end{cases} \implies \mathrm{Var}[\bar{X}] = 8.1 \text{ and } \Pr(|\bar{X} - \mu| < 2) = 0.1$$

Hoeffding's Inequality

Theorem: Hoeffding's Inequality

Suppose that $X_1, \ldots, X_n$ are distributed i.i.d, with $a \leq X_i \leq b$.
Then for any $\epsilon > 0$,

$$\Pr\left(\left|\bar{X} - \mathbb{E}[\bar{X}]\right| \geq \epsilon\right) \leq 2\exp\left(-\frac{2n\epsilon^2}{(b-a)^2}\right).$$

Equivalently, $\Pr\left(\left|\bar{X} - \mathbb{E}[\bar{X}]\right| \leq (b-a)\sqrt{\frac{\ln(2/\delta)}{2n}}\right) \geq 1 - \delta.$

Chebyshev's Inequality

Theorem: **Chebyshev's Inequality**

Suppose that $X_1, \ldots, X_n$ are distributed i.i.d. with variance σ^2.

Then for any $\epsilon > 0$,

$$\Pr\left(\left|\bar{X} - \mathbb{E}[\bar{X}]\right| \geq \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2}.$$

Equivalently, $\Pr\left(\left|\bar{X} - \mathbb{E}[\bar{X}]\right| \leq \sqrt{\frac{\sigma^2}{\delta n}}\right) \geq 1 - \delta.$

When to Use Chebyshev, When to Use Hoeffding?

- If $a \leq X_i \leq b$, then $\text{Var}[X_i] \leq \frac{1}{4}(b-a)^2$
- Hoeffding's inequality gives $\epsilon = (b-a)\sqrt{\frac{\ln(2/\delta)}{2n}} = \sqrt{\frac{\ln(2/\delta)}{2}}(b-a)\sqrt{\frac{1}{n}}$;

  Chebyshev's inequality gives $\epsilon = \sqrt{\frac{\sigma^2}{\delta n}} \leq \sqrt{\frac{(b-a)^2}{4\delta n}} = \frac{1}{2\sqrt{\delta}}(b-a)\sqrt{\frac{1}{n}}$
- **Hoeffding's inequality** gives a **tighter bound***, but it can only be used on **bounded** random variables
  - * whenever $\sqrt{\frac{\ln(2/\delta)}{2}} < \frac{1}{2\sqrt{\delta}} \iff \delta < \sim 0.232$
- **Chebyshev's inequality** can be applied even for **unbounded** variables

Sample Complexity

Definition:
The **sample complexity** of an estimator is the number of samples required to guarantee an error of at most ϵ with probability $1 - \delta$, for given δ and ϵ.

- We want sample complexity to be small
- Sample complexity is determined by:
  1. The **estimator** itself
     - Smarter estimators can sometimes improve sample complexity
  2. Properties of the **data generating process**
     - If the data are high-variance, we need more samples for an accurate estimate
     - But we can reduce the sample complexity if we can **bias** our estimate **toward the correct value**

Sample Complexity

Definition:
The **sample complexity** of an estimator is the number of samples required to guarantee an expected error of at most ϵ with probability $1 - \delta$, for given δ and ϵ.

For $\delta = 0.05$, **Chebyshev** gives

$$\epsilon = \sqrt{\frac{\sigma^2}{\delta n}} = \frac{1}{\sqrt{0.05}} \frac{\sigma}{\sqrt{n}}$$

$$\iff \quad \epsilon = 4.47 \frac{\sigma}{\sqrt{n}}$$

$$\iff \quad \sqrt{n} = 4.47 \frac{\sigma}{\epsilon}$$

$$\iff \quad n = 19.98 \frac{\sigma^2}{\epsilon^2}$$

With **Gaussian assumption** and $\delta = 0.05$,

$$\epsilon = 1.96 \frac{\sigma}{\sqrt{n}}$$

$$\iff \quad \sqrt{n} = 1.96 \frac{\sigma}{\epsilon}$$

$$\iff \quad n = 3.84 \frac{\sigma^2}{\epsilon^2}$$

Exercise: Sample Complexity for a Biased Estimator

- The concentration inequalities only tell us how the estimator concentrates around it's mean
- But if it is biased, then the mean of the estimator $\neq$ the true mean
- We can reduce the sample complexity (by reducing variance and/or by making stronger assumptions), but need to be careful about how much bias we introduce

Consistency of Downward-biased Mean Estimation

Example: $Y = \frac{1}{n+100} \sum_{i=1}^{n} X_i$

This estimator is **biased**:

$$\text{Bias}(Y) = \frac{n}{n+100}\mu - \mu = \frac{-100}{n+100}\mu$$

This estimator has **low variance**:

$$\text{Var}(Y) = \frac{n}{(n+100)^2}\sigma^2$$

Does this estimator have lower sample complexity than the sample average?

Is this estimator consistent?
(Namely, in the limit of samples, does it approach the true mean?)
(In other words, does it's bias go to zero?)

Summary

- **Concentration inequalities** let us bound the probability of a given estimator being at least ϵ from the estimated quantity
- **Sample complexity** is the **number of samples** needed to attain a desired error bound ϵ at a desired probability $1 - \delta$
- The **mean squared error** of an estimator **decomposes** into **bias** (squared) and **variance**
- Using a **biased** estimator can have **lower error** than an unbiased estimator
  - Bias the estimator based on some **prior information**
  - *But this only helps if the prior information is **correct,** cannot reduce error by adding in arbitrary bias*