Probability Theory

CMPUT 267: Basics of Machine Learning

§2.1-2.2

Recap

This class is about **understanding** machine learning techniques by understanding their basic **mathematical underpinnings**

- Assignment 1 released
- Thought Questions 1 due very soon (September 16)
  - Biggest reading since it covers much of the background
- Lab on Zoom today from 5 - 7 pm, to get started on Julia tutorial
- My office hours this week from 11 am - noon on Wednesday
  - will usually be 10 am - 11 am on Wednesday
- Some typos in the notes, updated on the website

Outline

1. Probabilities
2. Defining Distributions
3. Random Variables

Why Probabilities?

Even if the world is completely deterministic, outcomes can **look random** (**why?**)

Example: A high-tech gumball machine behaves according to $f(x_1, x_2) =$ output candy if x_1 & x_2, where x_1 = **has candy** and x_2 = **battery charged**.

- You can only see if it has candy (only see x_1)
- From your perspective, when $x_1 = 1$, sometimes candy is output, sometimes it isn't
- It **looks stochastic**, because it depends on the hidden input x_2

Measuring Uncertainty

- **Probability** is a way of **measuring** uncertainty
- We assign a number between 0 and 1 to **events** (hypotheses):
    - **0** means absolutely certain that statement is **false**
    - **1** means absolutely certain that statement is **true**
    - **Intermediate** values mean more or less certain
- Probability is a measurement of **uncertainty**, **not truth**
    - A statement with probability .75 is not "mostly true"
    - Rather, we **believe** it is more **likely** to be true than not

Subjective vs. Objective: The Frequentist Perspective

- Probabilities can be interpreted
  as **objective** statements about the **world**, or
  as **subjective** statements about an agent's **beliefs**.
- Objective view is called **frequentist:**
  - The probability of an event is the proportion of times it would happen **in the long run** of **repeated experiments**
  - Every event has a single, **true** probability

Subjective vs. Objective: The Bayesian Perspective

- Probabilities can be interpreted
  as **objective** statements about the **world**, or
  as **subjective** statements about an agent's **beliefs**.

- Subjective view is called **Bayesian:**
  - The probability of an event is a measure of an agent's **belief** about its likelihood
  - Different agents can legitimately have **different beliefs**, so they can legitimately assign **different probabilities** to the same event
  - Different beliefs due to different contexts and different assumptions

Example

- Estimating the average height of a person in the world
- There is a true population mean h
  - which can be computed by averaging the heights of every person
- An objective view is to directly estimate this true mean using data
  - e.g., compute a sample average $\bar{h}$ from a subpopulation by randomly sampling 1000 people from around the whole world
  - $\bar{h}$ estimates this true fact about the world, the true mean

Example

- There is a true population mean h
- An objective view is to directly estimate this true mean using data
  - e.g., compute a sample average $\bar{h}$ from a subpopulation by randomly sampling 1000 people from around the whole world
  - $\bar{h}$ estimates this true fact about the world, the true mean
- A subjective view is to maintain a belief $\bar{H}$ of what you believe is h
  - maintain probabilities $p(\bar{H})$ over plausible values of the average height

This distinction is a tad pedantic

- All you need to know is that we will both be trying to estimate underlying parameters (e.g., average heights)
- And we will reason about our own beliefs (uncertainty) for our estimates
- In math, we will sometimes directly compute sample averages and sometimes we will keep distributions of plausible values
  - They are both useful, with different preferences depending on the setting
- The one key thing to take away: **probabilities aren't always objectively about the world. We use them to reason about our own knowledge**

Prerequisites Check

- Derivatives
  - Rarely integration
  - I will teach you about partial derivatives
- Vectors and dot-products
- Set notation
  - Complement A^c of a set, union $A \cup B$ of sets, intersection of sets $A \cap B$
  - Set of sets, power set $\mathscr{P}(A)$
- Some exposure to probability. (We will cover much more today)

Terminology

- If you are unsure, notation sheet in the notes is a good starting point
- **Countable:** A set whose elements can be assigned an integer index
  - The integers themselves
  - Any finite set, e.g., $\{0.1, 2.0, 3.7, 4.123\}$
  - We'll sometimes say **discrete**, even though that's a little imprecise
- **Uncountable:** Sets whose elements *cannot* be assigned an integer index
  - Real numbers $\mathbb{R}$
  - Intervals of real numbers, e.g., $[0,1]$, $(-\infty, 0)$
  - Sometimes we'll say **continuous**

Outcomes and Events

All probabilities are defined with respect to a **measurable space** $(\Omega, \mathscr{E})$ of **outcomes** and **events**:

- Ω is the **sample space**: The set of all possible outcomes
- $\mathscr{E} \subseteq \mathscr{P}(\Omega)$ is the **event space**: A set of subsets of Ω that satisfies two key properties (that I will define in two slides)

Examples of Discrete & Continuous Sample Spaces and Events

Discrete (countable) outcomes

$\Omega = \{1,2,3,4,5,6\}$

$\Omega = \{\text{person, woman, man, camera, TV}, \ldots\}$

$\Omega = \mathbb{N}$

Continuous (uncountable) outcomes

$\Omega = [0,1]$

$\Omega = \mathbb{R}$

$\Omega = \mathbb{R}^k$

Event Spaces

Definition:
A set $\mathscr{E} \subseteq \mathscr{P}(\Omega)$ is an **event space** if it satisfies

1. $A \in \mathscr{E} \implies A^c \in \mathscr{E}$
2. $A_1, A_2, \ldots \in \mathscr{E} \implies \bigcup_{i=1}^{\infty} A_i \in \mathscr{E}$

1. A collection of outcomes (e.g., either a 2 or a 6 were rolled) is an event.
2. If we can measure that an event has occurred, then we should also be able to measure that the event has not occurred; i.e., its **complement** is measurable.
3. If we can measure two events separately, then we should be able to tell if one of them has happened; i.e., their **union** should be measurable too.

Examples of Discrete & Continuous Sample Spaces and Events

Discrete (countable) outcomes

$\Omega = \{1,2,3,4,5,6\}$

$\Omega = \{\text{person, woman, man, camera, TV}, \ldots\}$

$\Omega = \mathbb{N}$

$\mathscr{E} = \{\emptyset, \{1,2\}, \{3,4,5,6\}, \{1,2,3,4,5,6\}\}$

Typically**:** $\mathscr{E} = \mathscr{P}(\Omega)$

Powerset is the set of all subsets

Continuous (uncountable) outcomes

$\Omega = [0,1]$

$\Omega = \mathbb{R}$

$\Omega = \mathbb{R}^k$

$\mathscr{E} = \{\emptyset, [0,0.5], (0.5,1.0], [0,1]\}$

Typically: $\mathscr{E} = B(\Omega)$ ("Borel field")

Borel field is the set of all subsets of non-neglible size (e.g., intervals $[0.1, 0.1 + \epsilon]$)

Discrete vs. Continuous Sample Spaces

Discrete (countable) outcomes

$\Omega = \{1,2,3,4,5,6\}$

$\Omega = \{\text{person, woman, man, camera, TV}, \ldots\}$

$\Omega = \mathbb{N}$

$\mathscr{E} = \{\emptyset, \{1,2\}, \{3,4,5,6\}, \{1,2,3,4,5,6\}\}$

Typically**:** $\mathscr{E} = \mathscr{P}(\Omega)$

Question:
$\mathscr{E} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$?

Continuous (uncountable) outcomes

$\Omega = [0,1]$

$\Omega = \mathbb{R}$

$\Omega = \mathbb{R}^k$

$\mathscr{E} = \{\emptyset, [0,0.5], (0.5,1.0], [0,1]\}$

Typically: $\mathscr{E} = B(\Omega)$ ("Borel field")

Note: *not* $\mathscr{P}(\Omega)$

Exercise

- Write down the power set of {1, 2, 3}
- More advanced: Why is the power set a valid event space? Hint: Check the two properties

Definition:

A set $\mathcal{E} \subseteq \mathcal{P}(\Omega)$ is an **event space** if it satisfies

1. $A \in \mathcal{E} \implies A^c \in \mathcal{E}$
2. $A_1, A_2, \ldots \in \mathcal{E} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{E}$

Exercise answer

- $\Omega = \{1,2,3\}$
- $\mathscr{P}(\Omega) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$
- Proof that the power set satisfies the two properties
- Take any $A \in \mathscr{P}(\Omega)$ (e.g., $A = \{1\}$ or $A = \{1,2\}$). Then $A^c = \Omega - A$ is a subset of Ω, and so $A^c \in \mathscr{P}(\Omega)$ since the power set contains all subsets
- Take any $A, B \in \mathscr{P}(\Omega)$. Then $A \cup B \subset \Omega$, and so $A \cup B \in \mathscr{P}(\Omega)$
- More generally, for an infinite union, see: https://proofwiki.org/wiki/Power_Set_is_Closed_under_Countable_Unions

Axioms

Definition:

Given a measurable space $(\Omega, \mathscr{E})$, any function $P : \mathscr{E} \to [0,1]$ satisfying

1. **unit measure:** $P(\Omega) = 1$, and
2. ***σ*-additivity:** $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ for any countable sequence $A_1, A_2, \ldots \in \mathscr{E}$ where $A_i \cap A_j = \varnothing$ whenever $i \neq j$

is a **probability measure** (or **probability distribution**).

If P is a probability measure over $(\Omega, \mathscr{E})$, then $(\Omega, \mathscr{E}, P)$ is a **probability space**.

Defining a Distribution

Example:

$\Omega = \{0,1\}$

$\mathscr{E} = \{\varnothing, \{0\}, \{1\}, \Omega\}$

$$P = \begin{cases} 1-\alpha & \text{if } A = \{0\} \\ \alpha & \text{if } A = \{1\} \\ 0 & \text{if } A = \varnothing \\ 1 & \text{if } A = \Omega \end{cases}$$

where $\alpha \in [0,1]$.

Questions:

1. Do you recognize this distribution?
2. How should we choose P in practice?
   a. Can we choose an arbitrary function?
   b. How can we guarantee that all of the constraints will be satisfied?

Probability Mass Functions (PMFs)

Definition: Given a **discrete** sample space Ω and event space $\mathscr{E} = \mathscr{P}(\Omega)$, any function $p : \Omega \to [0,1]$ satisfying $\sum_{\omega \in \Omega} p(\omega) = 1$ is a **probability mass function**.

- For a discrete sample space, instead of defining P directly, we can define a **probability mass function** $p : \Omega \to [0,1]$.
- p gives a probability for **outcomes** instead of **events**
- The probability for any event $A \in \mathscr{E}$ is then defined as $P(A) = \sum_{\omega \in A} p(\omega)$.

Example: PMF for a Fair Die

A **categorical distribution** is a distribution over a **finite** outcome space, where the probability of each outcome is specified separately.

Example: Fair Die

$$\Omega = \{1,2,3,4,5,6\}$$

$$p(\omega) = \frac{1}{6}$$

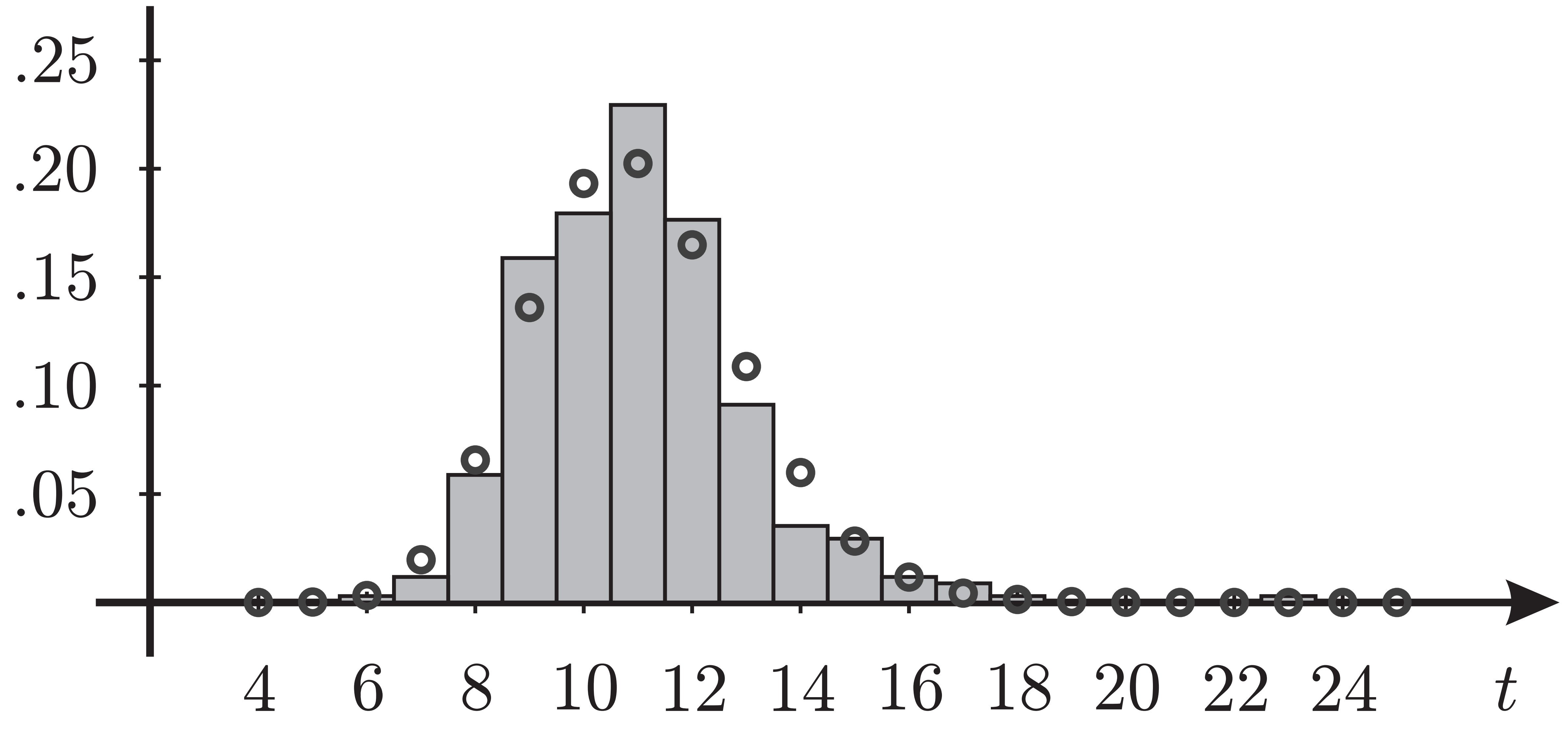
ω	$p(\omega)$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

Questions:

1. What is a possible event? What is its probability?
2. What is the event space?

Example: Using a PMF

- Suppose that you recorded your commute time (in minutes) every day for a year (i.e., 365 recorded times).
- **Question:** How do you get $p(t)$?
- **Question:** How is $p(t)$ useful?



Useful PMFs: Bernoulli

A **Bernoulli distribution** is a special case of a **categorical distribution** in which there are only two outcomes. It has a single **parameter** $\alpha \in (0,1)$.

$\Omega = \{T, F\}$ (or $\Omega = \{S, F\}$)

$$p(\omega) = \begin{cases} \alpha & \text{if } \omega = T \\ 1 - \alpha & \text{if } \omega = F \,. \end{cases}$$

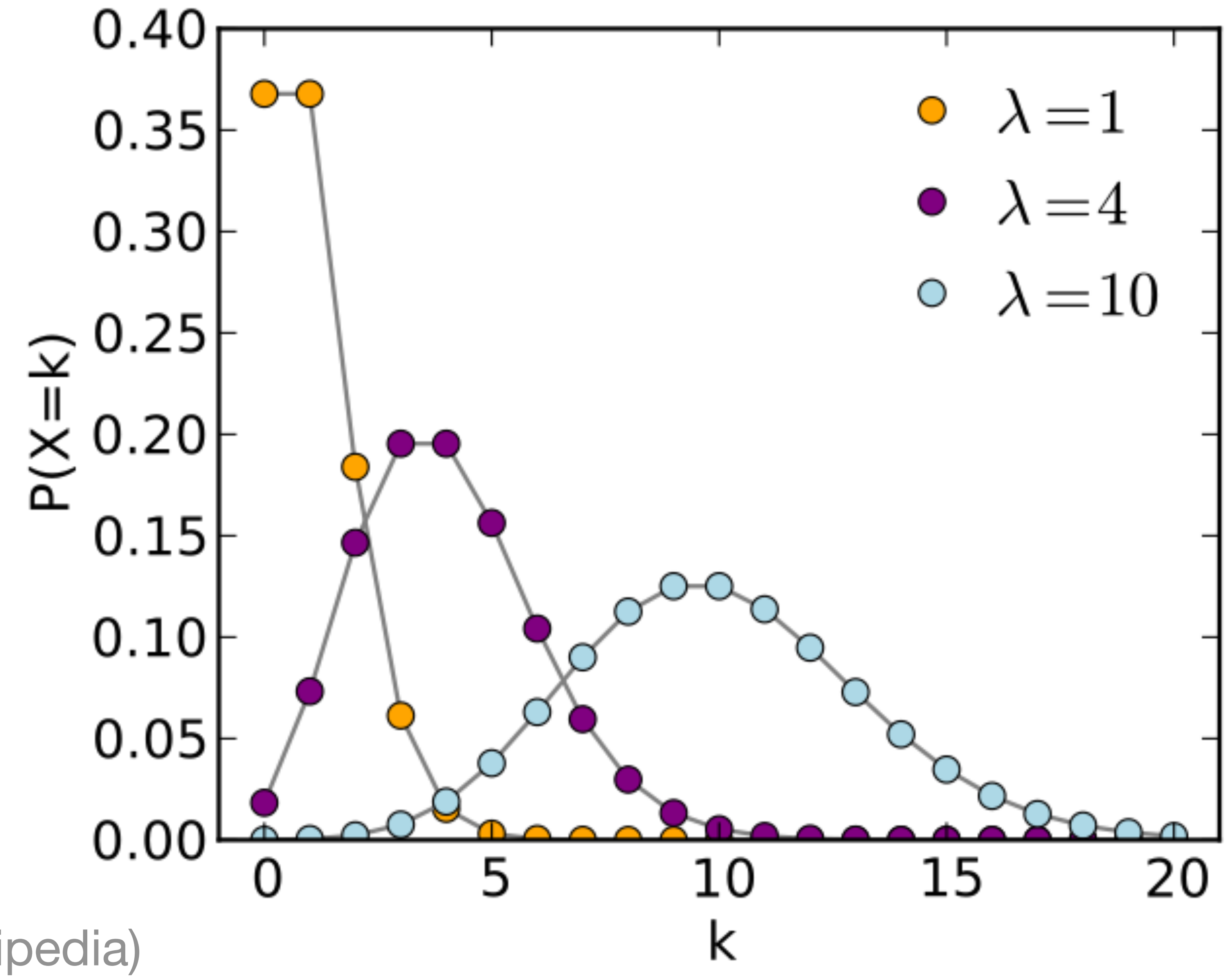
Alternatively: $\Omega = \{0,1\}$

$$p(k) = \alpha^k (1-\alpha)^{1-k} \quad \text{for } k \in \{0,1\}$$

Useful PMFs: Poisson

A **Poisson distribution** is a distribution over the non-negative integers.
It has a single parameter $\lambda \in (0,\infty)$.

E.g., number of calls received by a call centre in an hour, λ is the average number of calls



(Image: Wikipedia)

$$p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

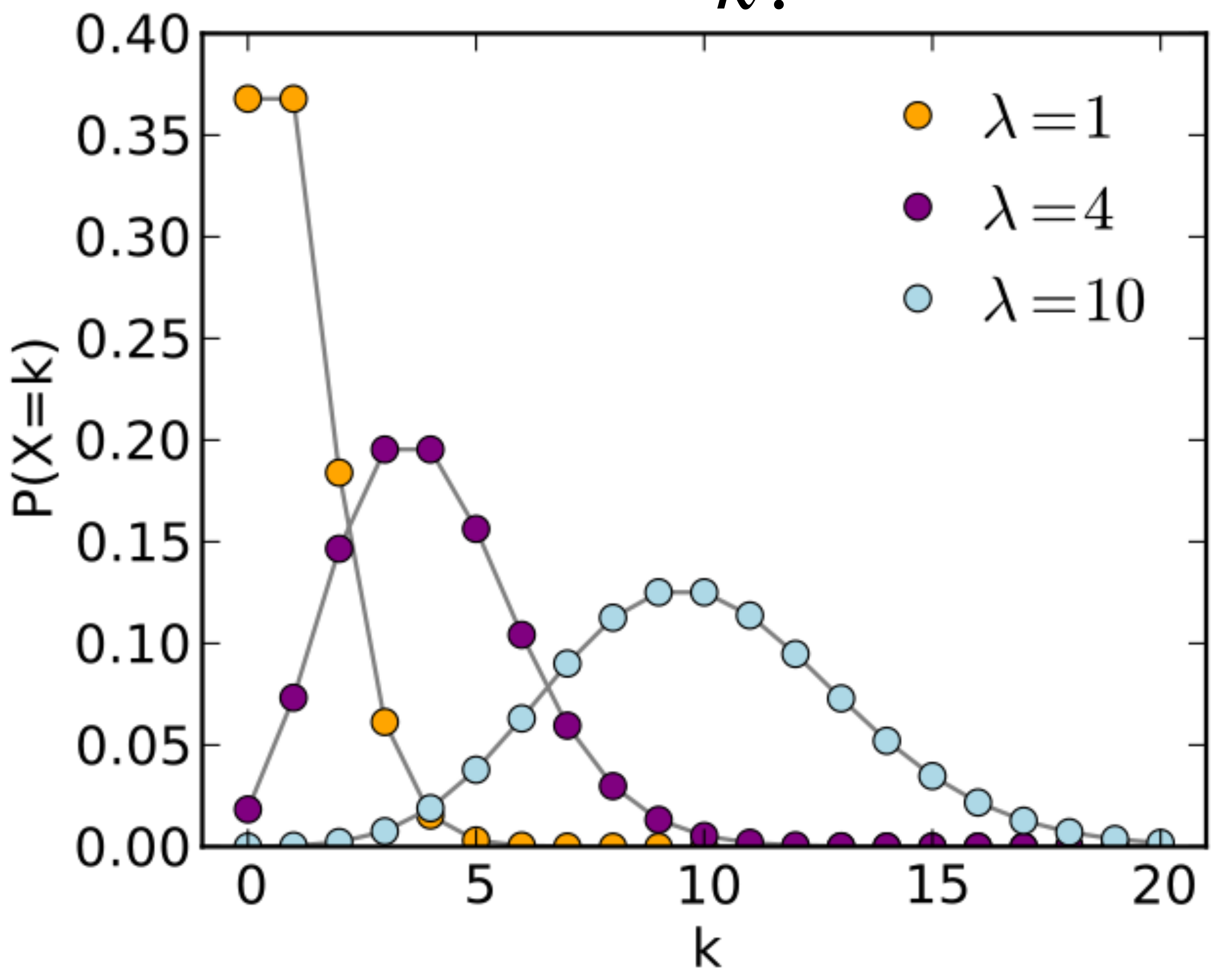
Questions:

1. Could we define this with a table instead of an equation?
2. How can we check whether this is a valid PMF?
3. λ real-valued, but outcome is discrete. What might be the mode (most likely outcome)?

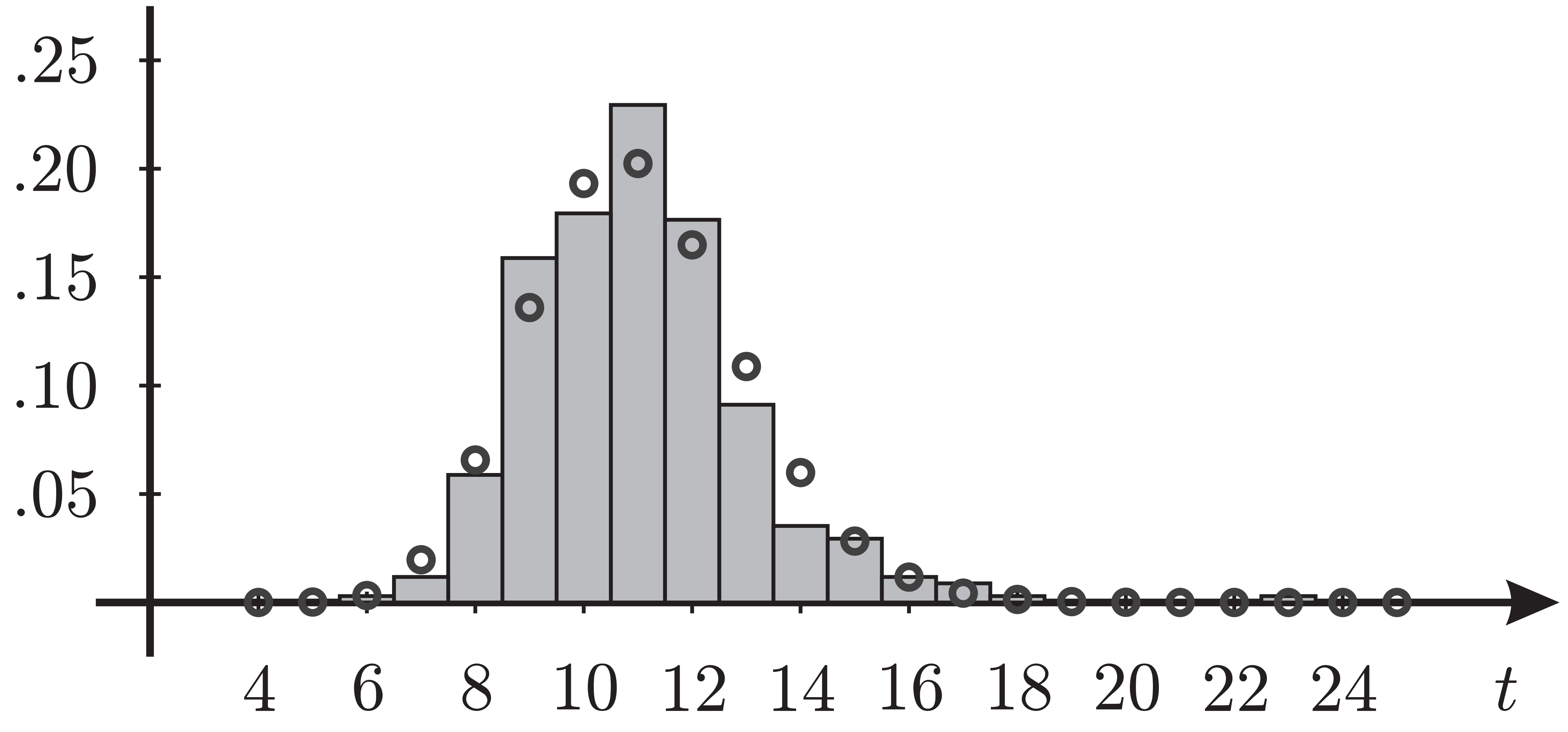
Commute Times Again

- **Question:** Could we use a **Poisson distribution** for commute times (instead of a categorical distribution)?
- **Question:** What would be the benefit of using a Poisson distribution?

$$p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

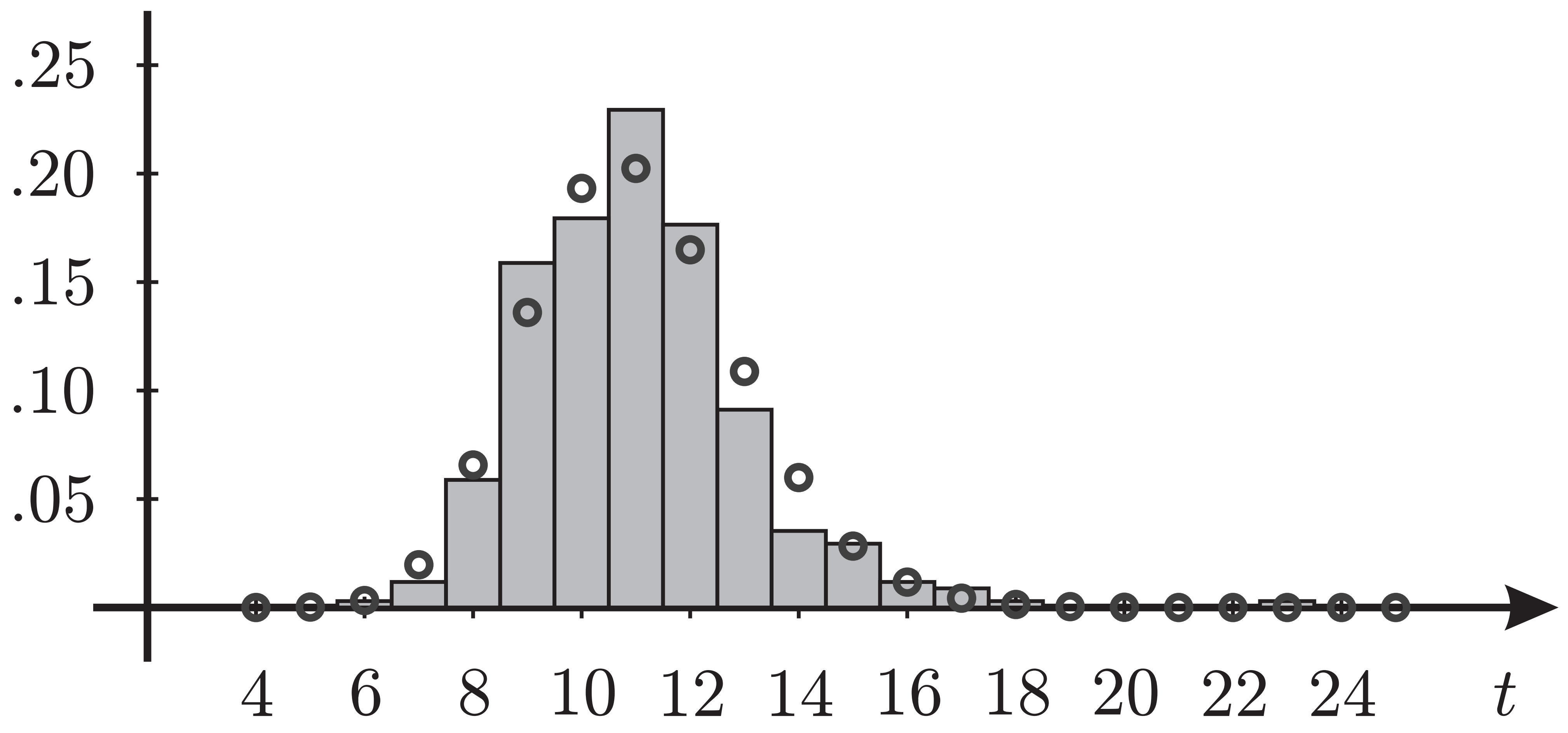


$$p(4) = 1/365,\ p(5) = 2/365,\ p(6) = 4/365,\ \ldots$$



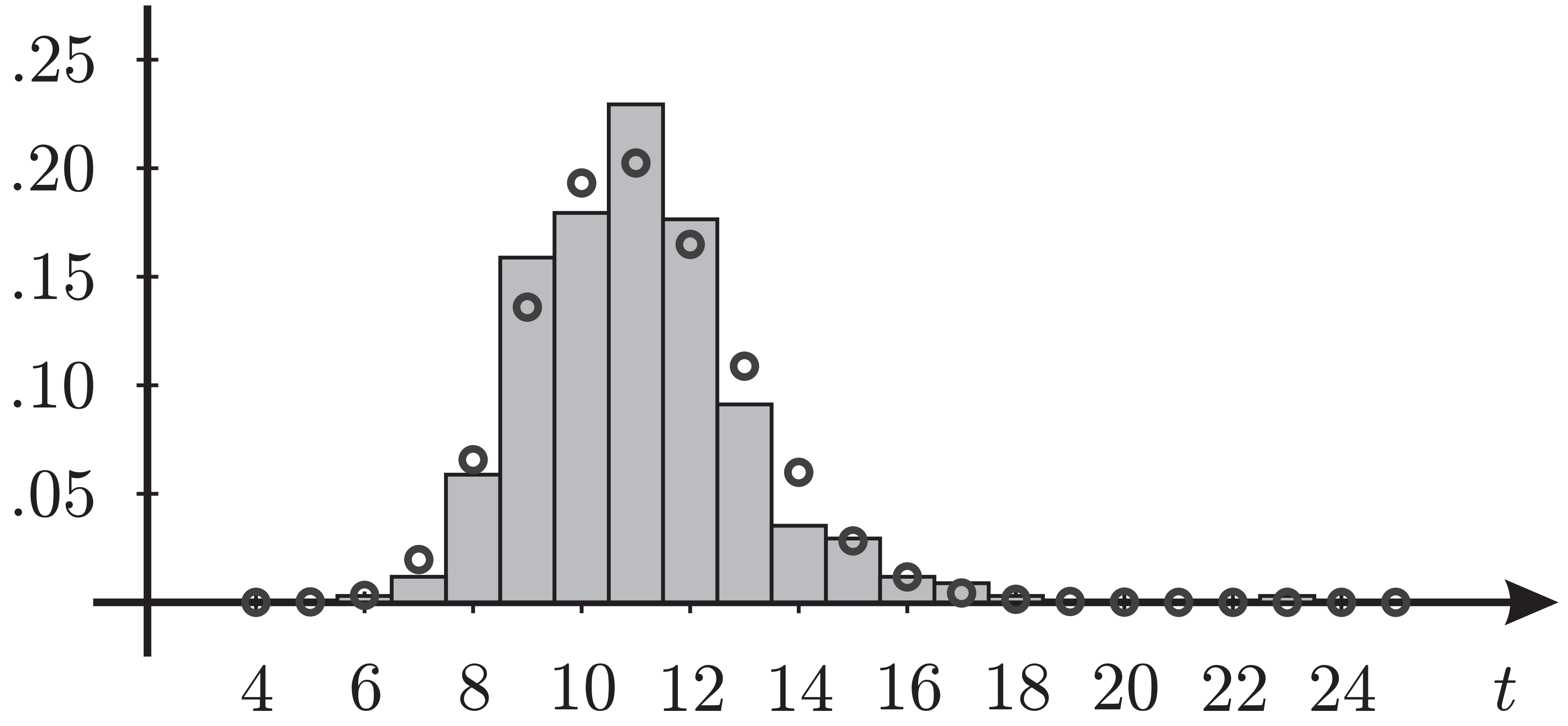
Continuous Commute Times

- It never actually takes *exactly* 12 minutes; I rounded each observation to the nearest integer number of minutes.
  - Actual data was 12.345 minutes, 11.78213 minutes, etc.



Using Histograms

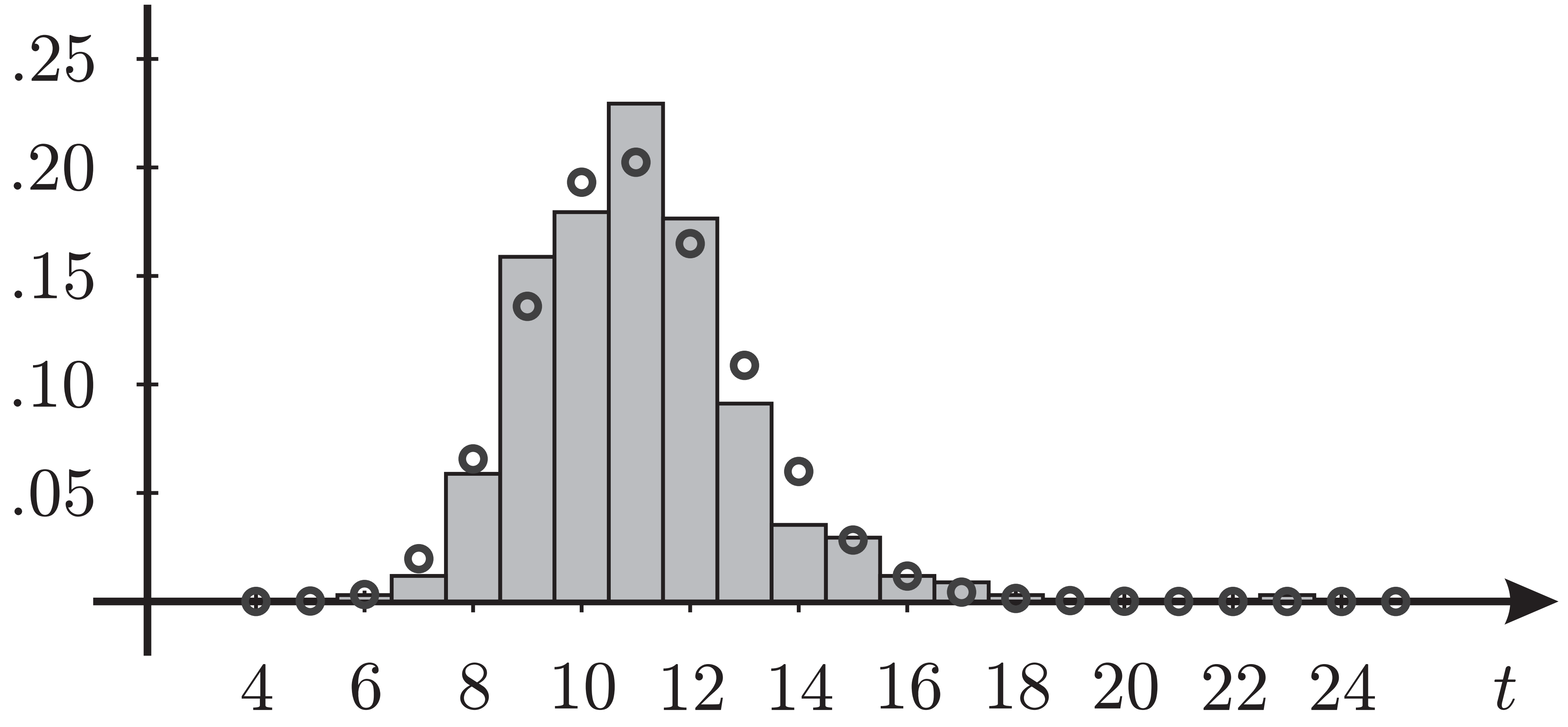
Consider the continuous commuting example again, with observations 12.345 minutes, 11.78213 minutes, etc.



- **Question:** How could we turn our observations into a histogram?
- **Question:** How do we use we the histogram to get these probabilities?

Continuous Commute Times

- It never actually takes *exactly* 12 minutes; I rounded each observation to the nearest integer number of minutes.
  - Actual data was 12.345 minutes, 11.78213 minutes, etc.
- **Question:** Could we use a Poisson distribution to predict the *exact* commute time (rather than the nearest number of minutes)? Why?



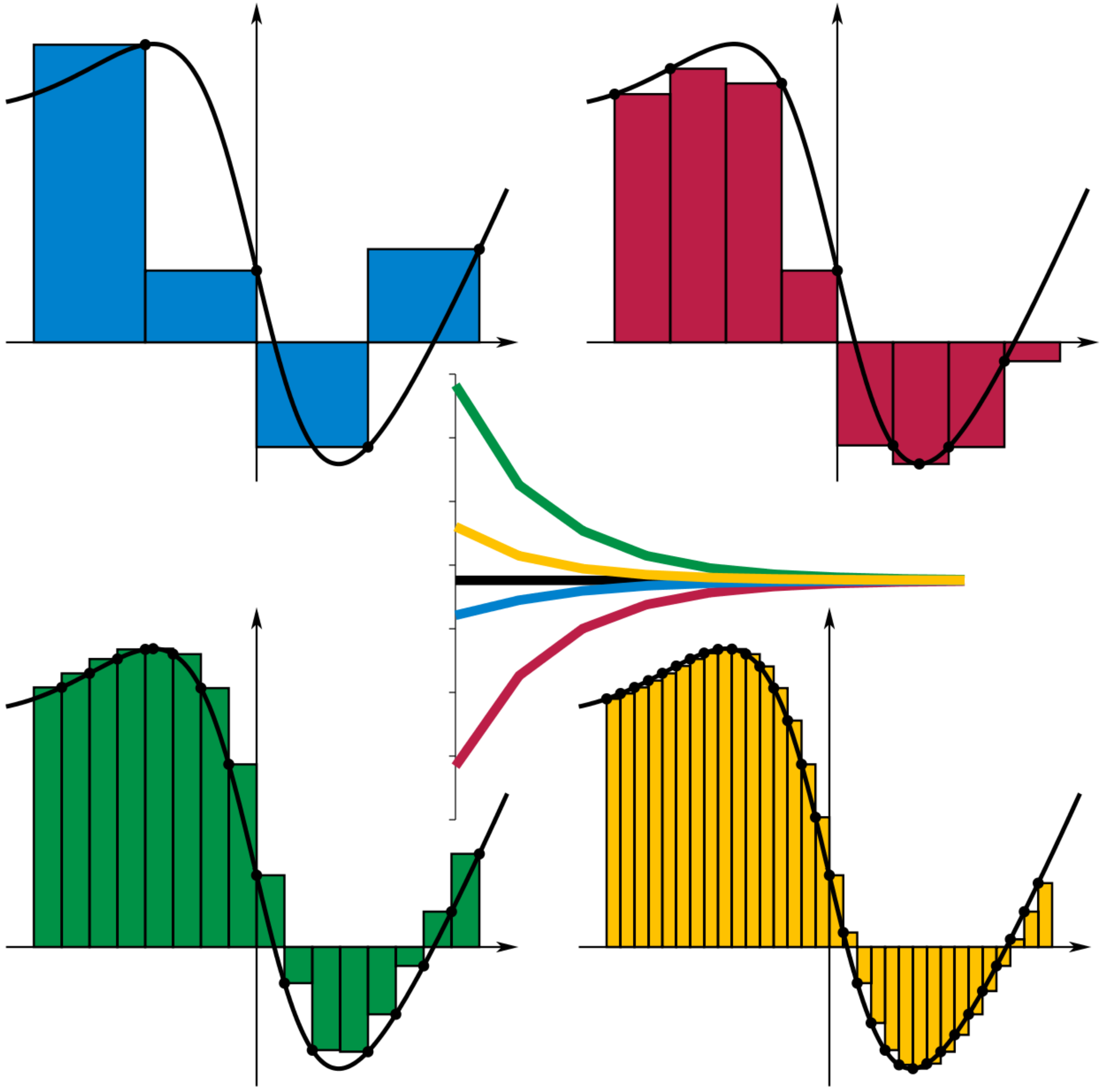
Probability Density Functions (PDFs)

Definition: Given a **continuous** sample space Ω and event space $\mathscr{E} = B(\Omega)$, any function $p : \Omega \to [0,\infty)$ satisfying $\int_\Omega p(\omega)d\omega = 1$ is a **probability density function**.

- For a continuous sample space, instead of defining P directly, we can define a **probability density function** $p : \Omega \to [0,\infty)$.
- The probability for any event $A \in \mathscr{E}$ is then defined as

$$P(A) = \int_A p(\omega)d\omega.$$

Recall Integration

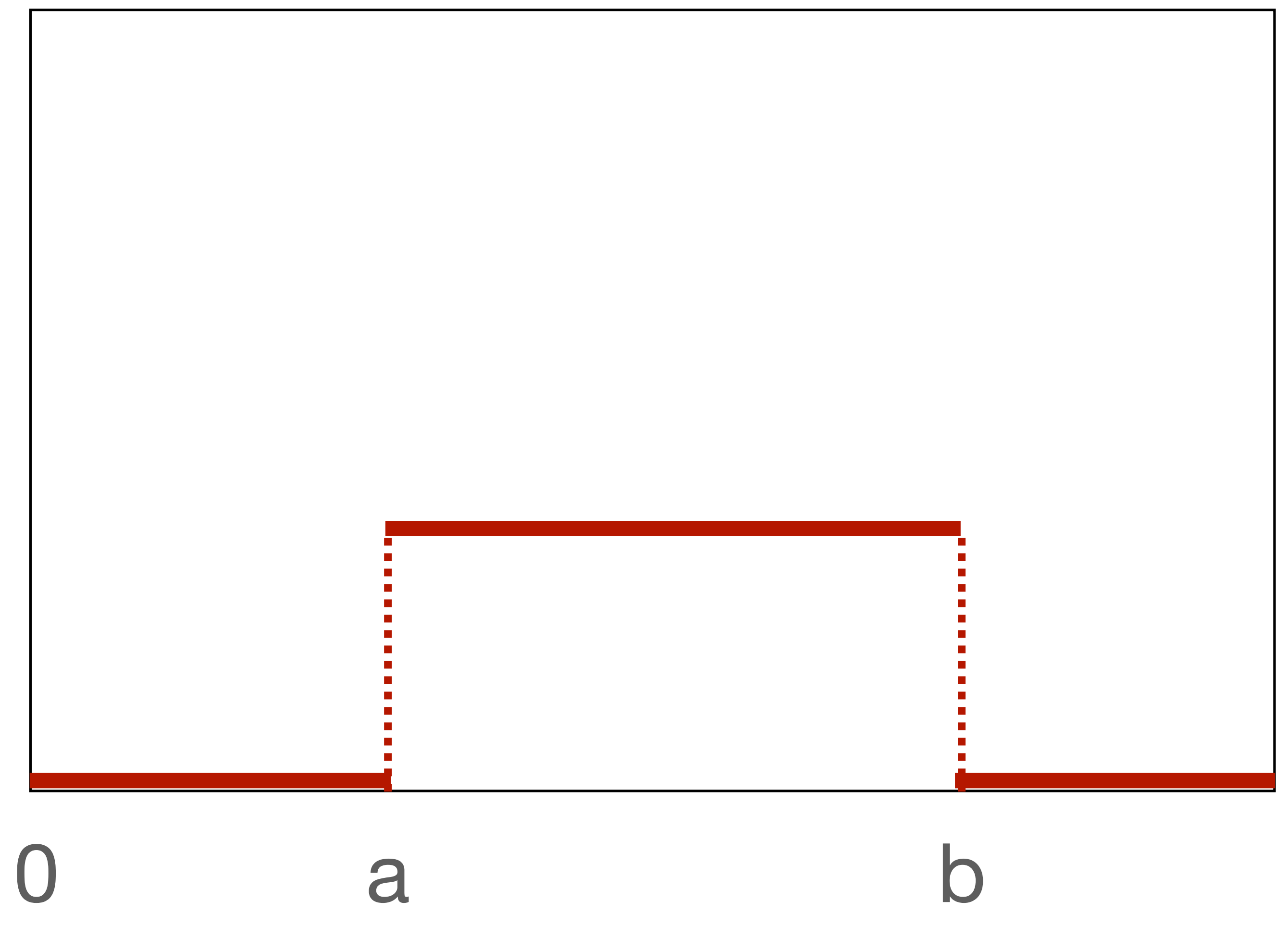


Useful PDFs: Uniform

A **uniform distribution** is a distribution over a real interval. It has two parameters: a and b.

$$\Omega = [a, b]$$

$$p(\omega) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq \omega \leq b, \\ 0 & \text{otherwise.} \end{cases}$$



Question: Does Ω have to be bounded?

Exercise: Check that the uniform pdf satisfies the required properties

Recall that the antiderivative of 1 is x, because the derivative of x is 1

$$\int_a^b p(x)dx = \int_a^b \frac{1}{b-a}dx$$

$$= \frac{1}{b-a}\int_a^b dx = \frac{1}{b-a}x\,|_a^b$$

$$= \frac{1}{b-a}(b-a) = 1$$

Useful PDFs: Gaussian

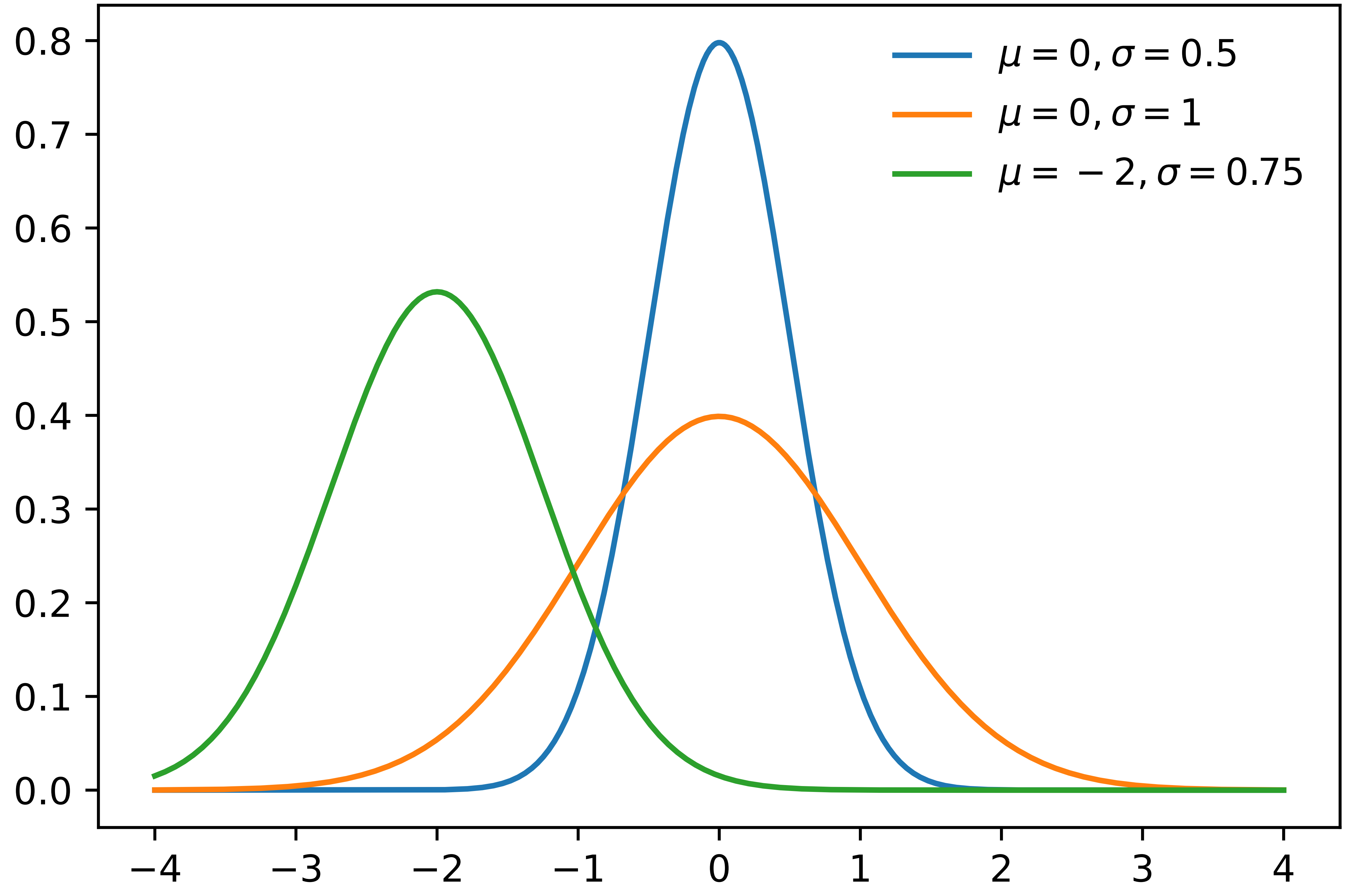
A **Gaussian distribution** is a distribution over the real numbers. It has two parameters: $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$.

$$\Omega = \mathbb{R}$$

$$p(\omega) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(\omega - \mu)^2\right)$$

where $\exp(x) = e^x$

Also called a normal distribution and written $\mathcal{N}(\mu, \sigma^2)$



Why the distinction between PMFs and PDFs?

1. When the sample space Ω is **discrete:**

$$P(A) = \sum_{\omega \in \Omega} p(\omega)$$

- Singleton event: $P(\{\omega\}) = p(\omega)$ for $\omega \in \Omega$

2. When the sample space Ω is **continuous:**

$$P(A) = \int_A p(\omega) d\omega$$

- Example: Stopping time for a car with $\Omega = [3,12]$
- **Question:** What is the probability that the stopping time is *exactly* 3.14159?

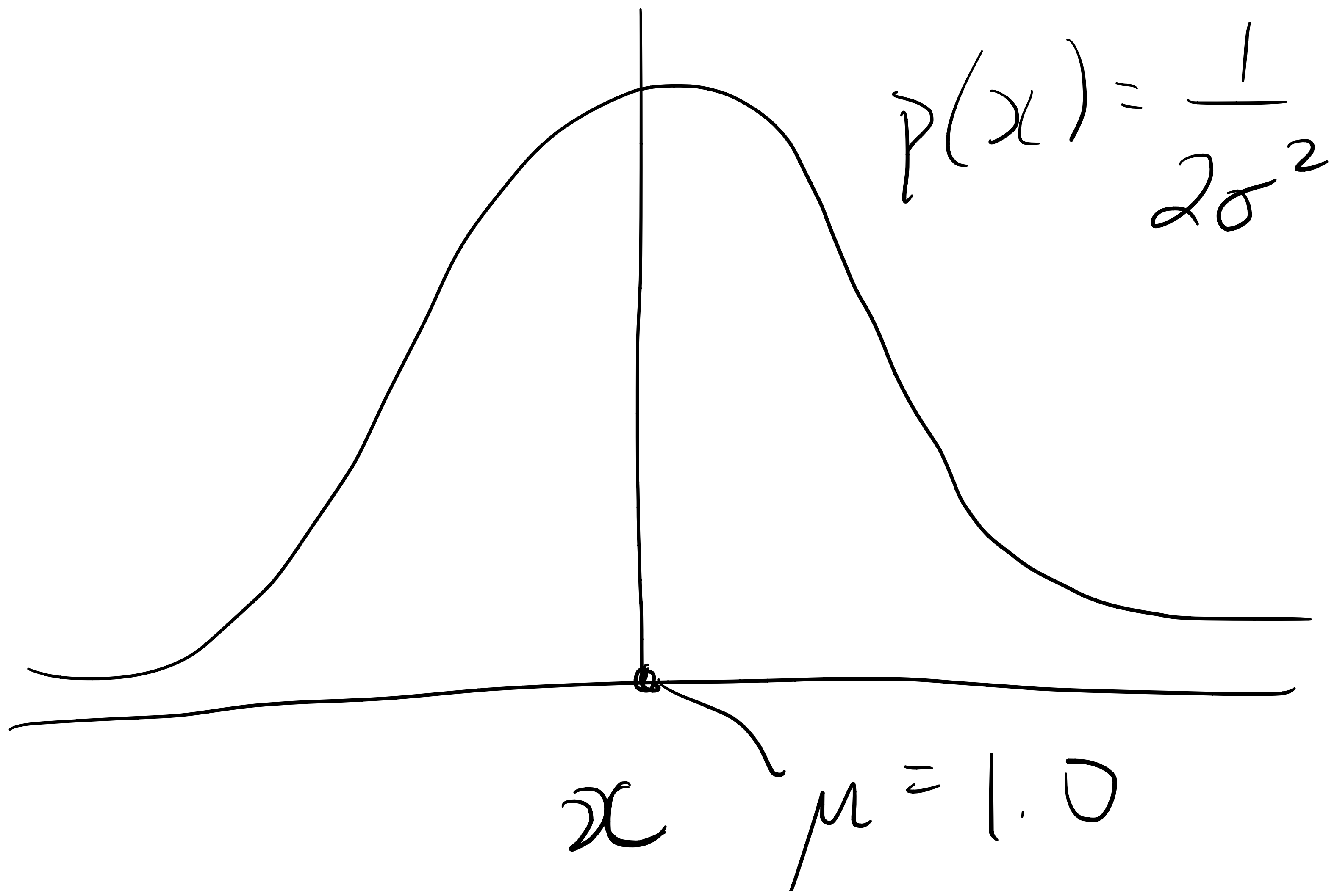
$$P(\{3.14159\}) = \int_{3.14159}^{3.14159} p(\omega) d\omega$$

- More reasonable: Probability that stopping time is between 3 to 3.5.

Example comparing integration and summation

Imagine we have a Gaussian distribution

$$p(x) = \frac{1}{2\sigma^2} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

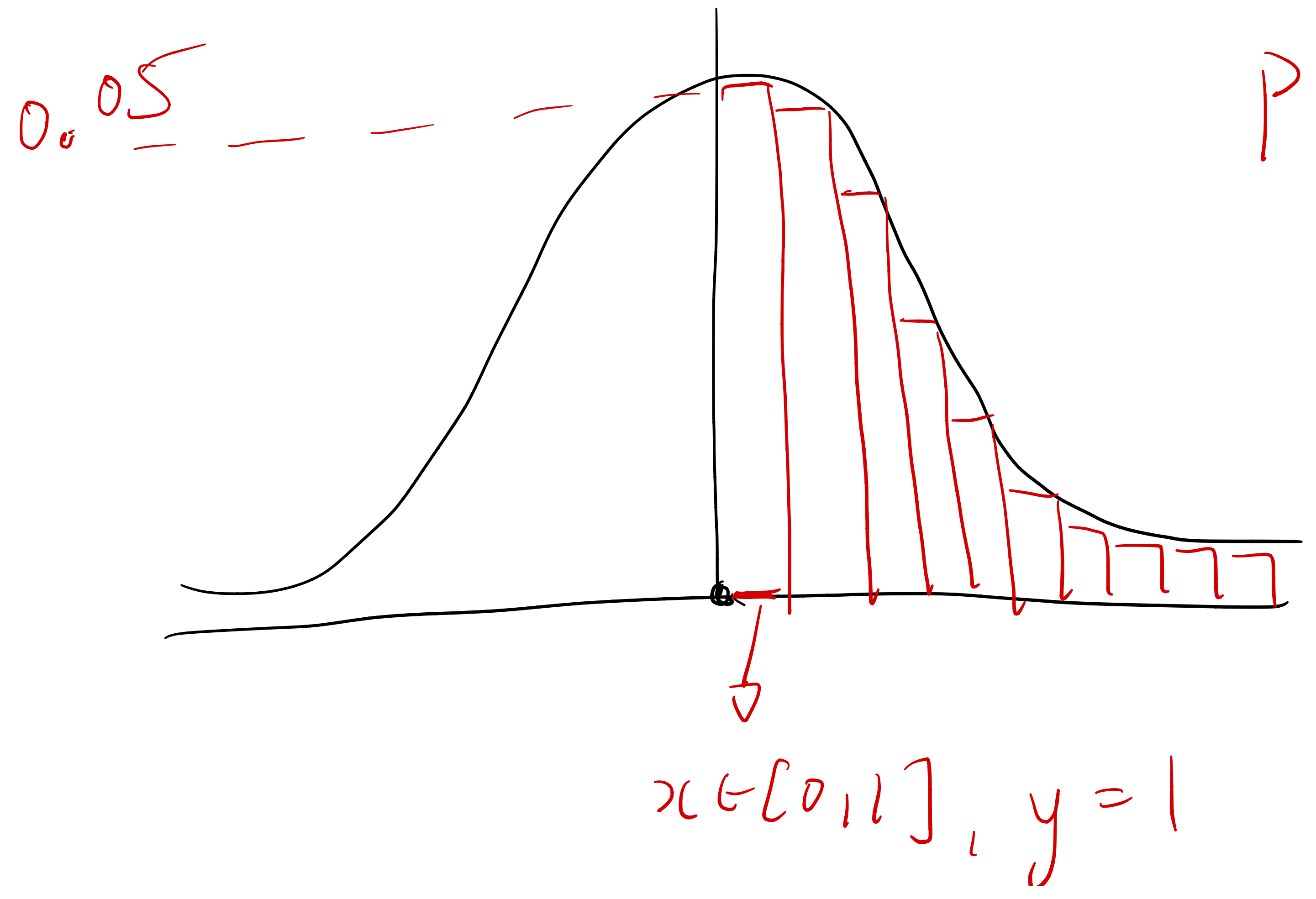


Example comparing integration and summation (cont)

Let's pretend we discretized to get a PMF

$y = i$ for $x \in (i-1, i]$

$p(y=1) = 0.05$



Example comparing integration and summation (cont)

Let's pretend we discretized to get a PMF

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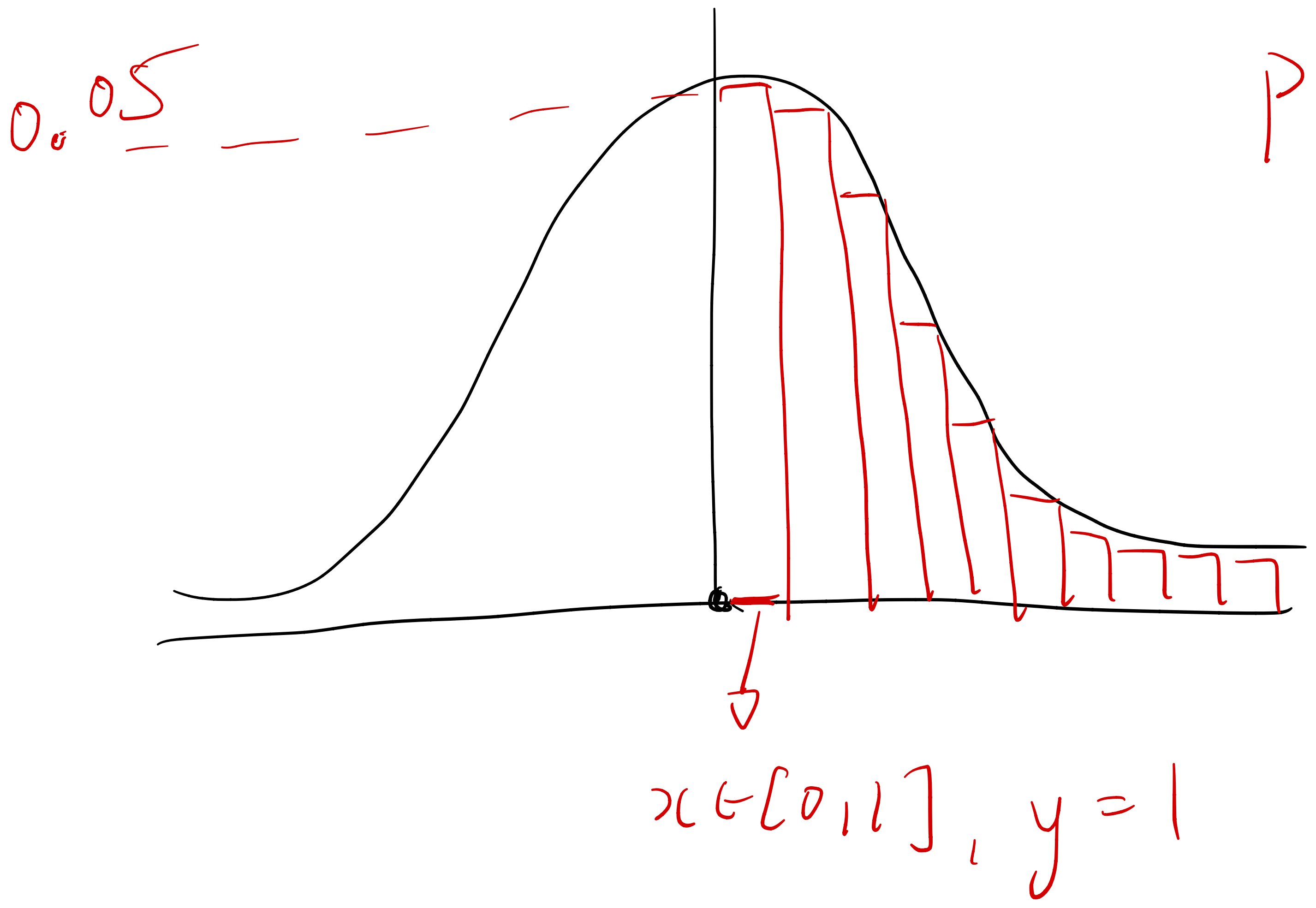


$p(y=1) = 0.05$

when we ask

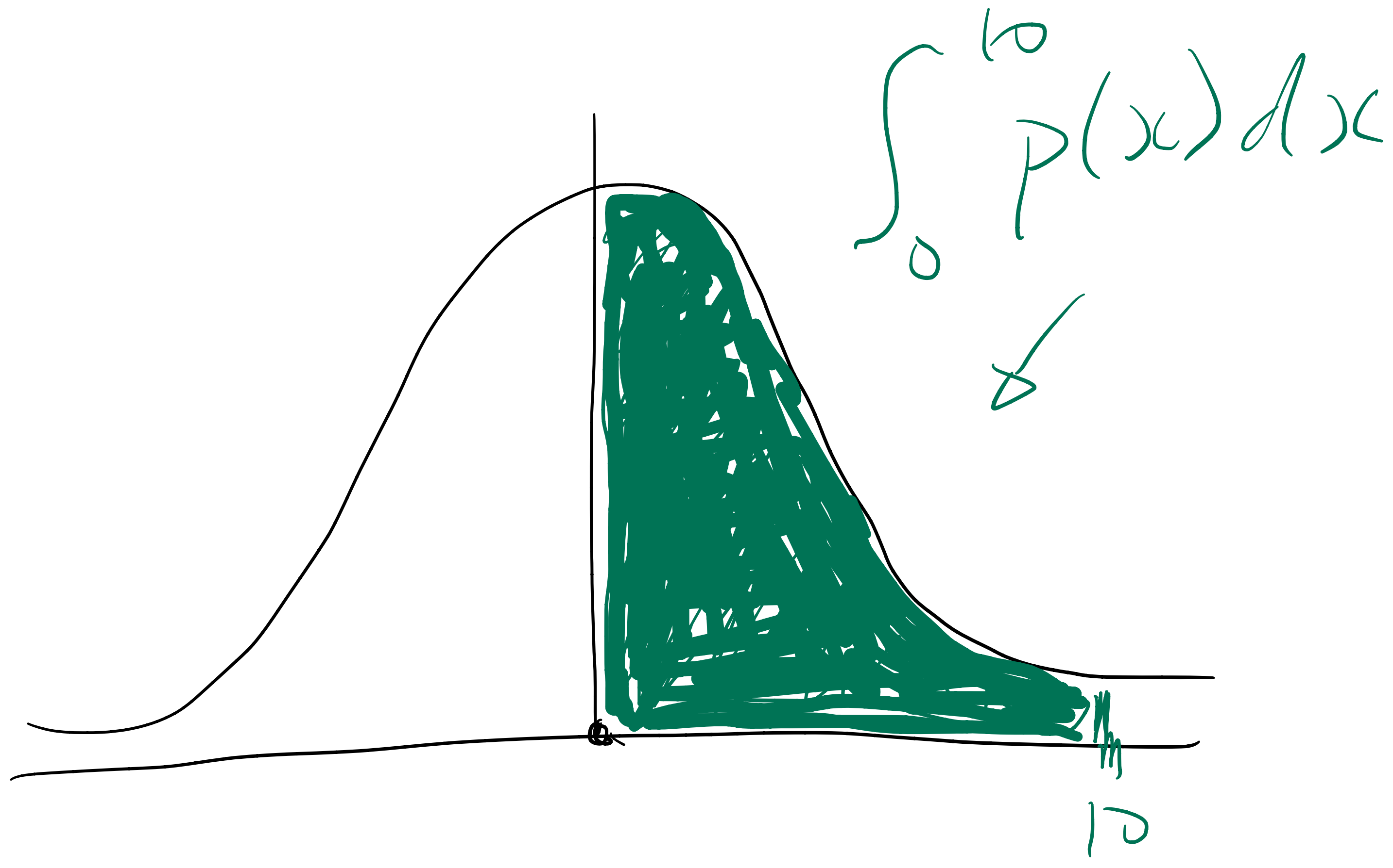
$$Pr(X \in [0,10]) = \int_0^{10} p(x)\,dx$$

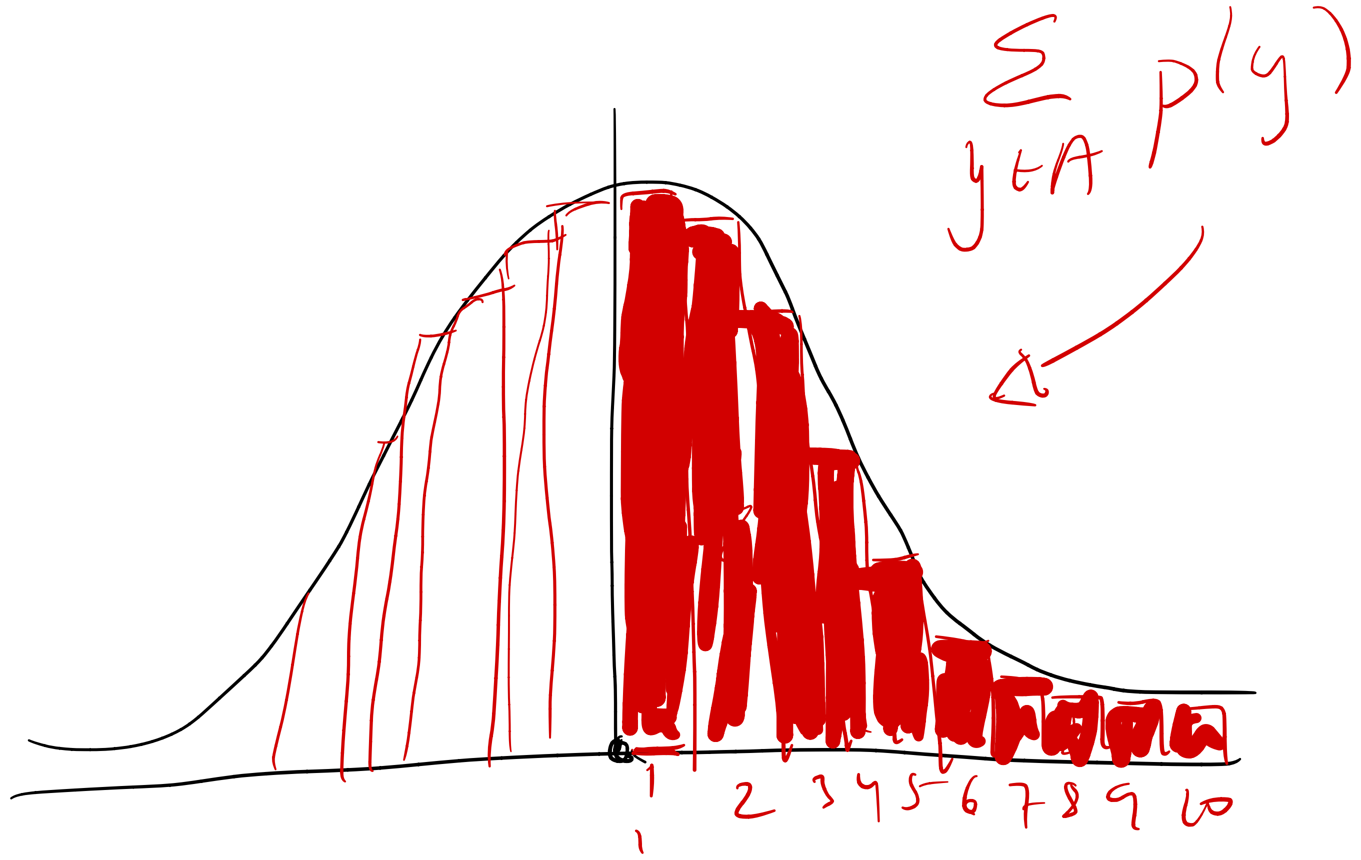
Similar to

$$Pr(Y \in \underbrace{\{1,2,3,\ldots,10\}}_{A}) = \sum_{y \in A} p(y)$$

Example comparing integration and summation (cont)

Both reflect density or mass in a region.



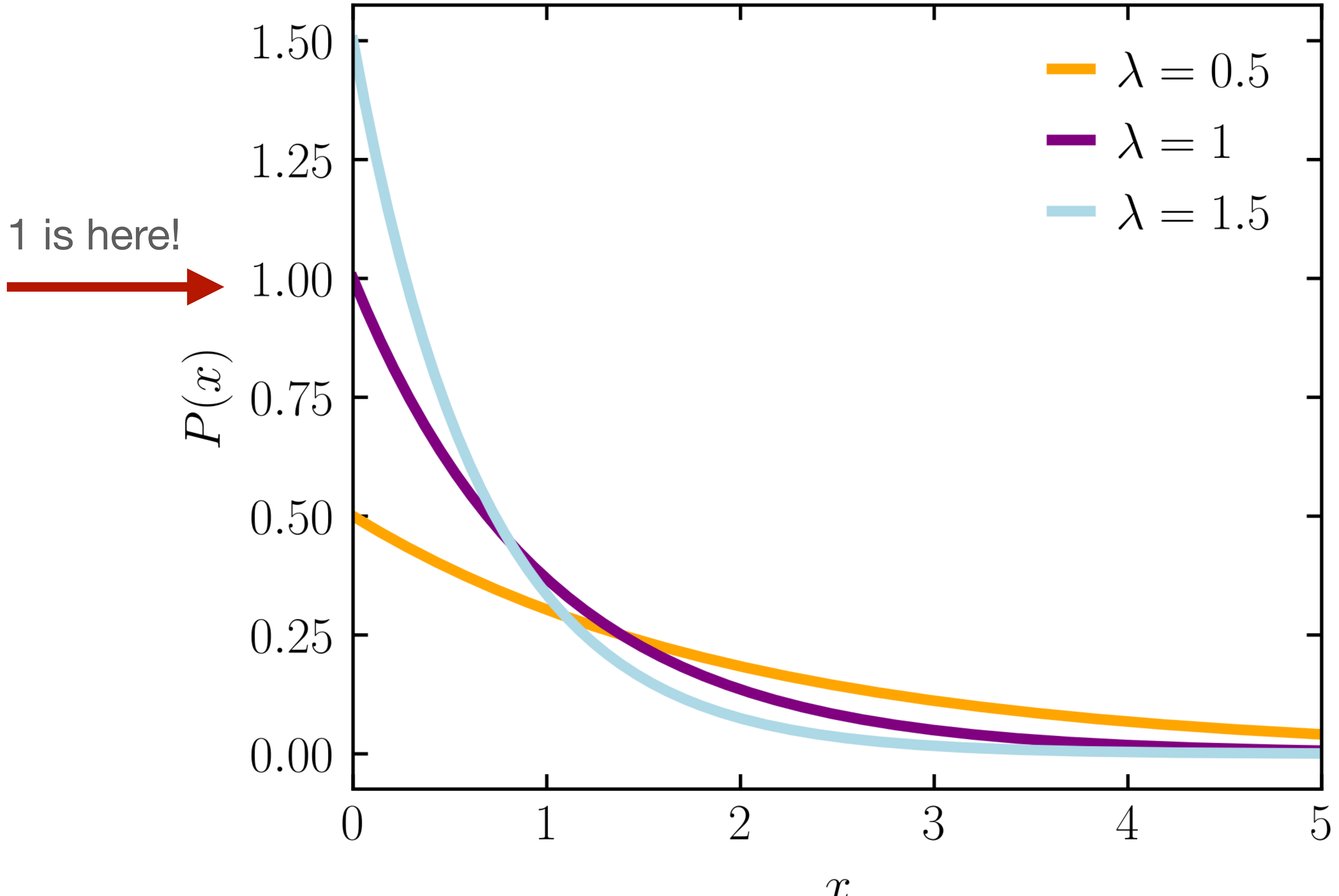


Useful PDFs: Exponential

An **exponential distribution** is a distribution over the positive reals. It has one parameter $\lambda > 0$.

$$\Omega = \mathbb{R}^+$$

$$p(\omega) = \lambda \exp(-\lambda\omega)$$



Why can the density be above 1?

Consider an interval event $A = [x, x + \Delta x]$, for small Δx.

$$P(A) = \int_{x}^{x+\Delta x} p(\omega)\, d\omega .$$

$$\approx p(x)\Delta x$$

- $p(x)$ can be big, because Δx can be very small
  - In particular, $p(x)$ can be bigger than 1
- But $P(A)$ **must** be less than or equal to 1

Review So Far

- Imagine I asked you to tell me the probability that my birthday is on February 10 or July 9.
  - What is the outcome space and what is the event for this question?
  - Would we use a PMF or PDF to model these probabilities?
- Imagine I asked you to tell me the probability that the uber would be here in between 3-5 minutes
  - What is the outcome space and what is the event for this question?
  - Would we use a PMF or PDF to model these probabilities?

Random Variables

Random variables are a way of reasoning about a complicated underlying probability space in a more straightforward way.

Example: Suppose we observe both a die's number, and where it lands.

$$\Omega = \{(left,1), (right,1), (left,2), (right,2), \ldots, (right,6)\}$$

We might want to think about the probability that we get a large number, without thinking about where it landed.

We could ask about $P(X \geq 4)$, where

X = number that comes up.

Random Variables, Formally

Given a probability space $(\Omega, \mathscr{E}, P)$, a **random variable** is a function $X : \Omega \to \Omega_X$ (where Ω_X is a new outcome space), satisfying

$$\{\omega \in \Omega \mid X(\omega) \in A\} \in \mathscr{E} \quad \forall A \in B(\Omega_X).$$

It follows that $P_X(A) = P(\{\omega \in \Omega \mid X(\omega) \in A\})$.

Example: Let Ω be a population of people, and $X(\omega)$ = height in cm, and the event $A = [150,170]$.

$$P(X \in A) = P(150 \le X \le 170) = P(\{\omega \in \Omega : X(\omega) \in A\}).$$

RVs are intuitive

- All the probability rules remain the same, since RVs are a mapping to create a new outcome space, event space and probabilities
- The notation may look onerous, but they simply formalize something we do naturally: specify the variable we care about, knowing it is defined by a more complex underlying distribution
- We have really already been talking about RVs
  - e.g., for $X =$ dice outcome, event $A = \{5,6\}$, $P(A) = P(X \geq 4)$
  - It is less cumbersome to talk about X and boolean expressions

Random Variables Simplify Terminology

- A Boolean expression involving random variables defines an event:

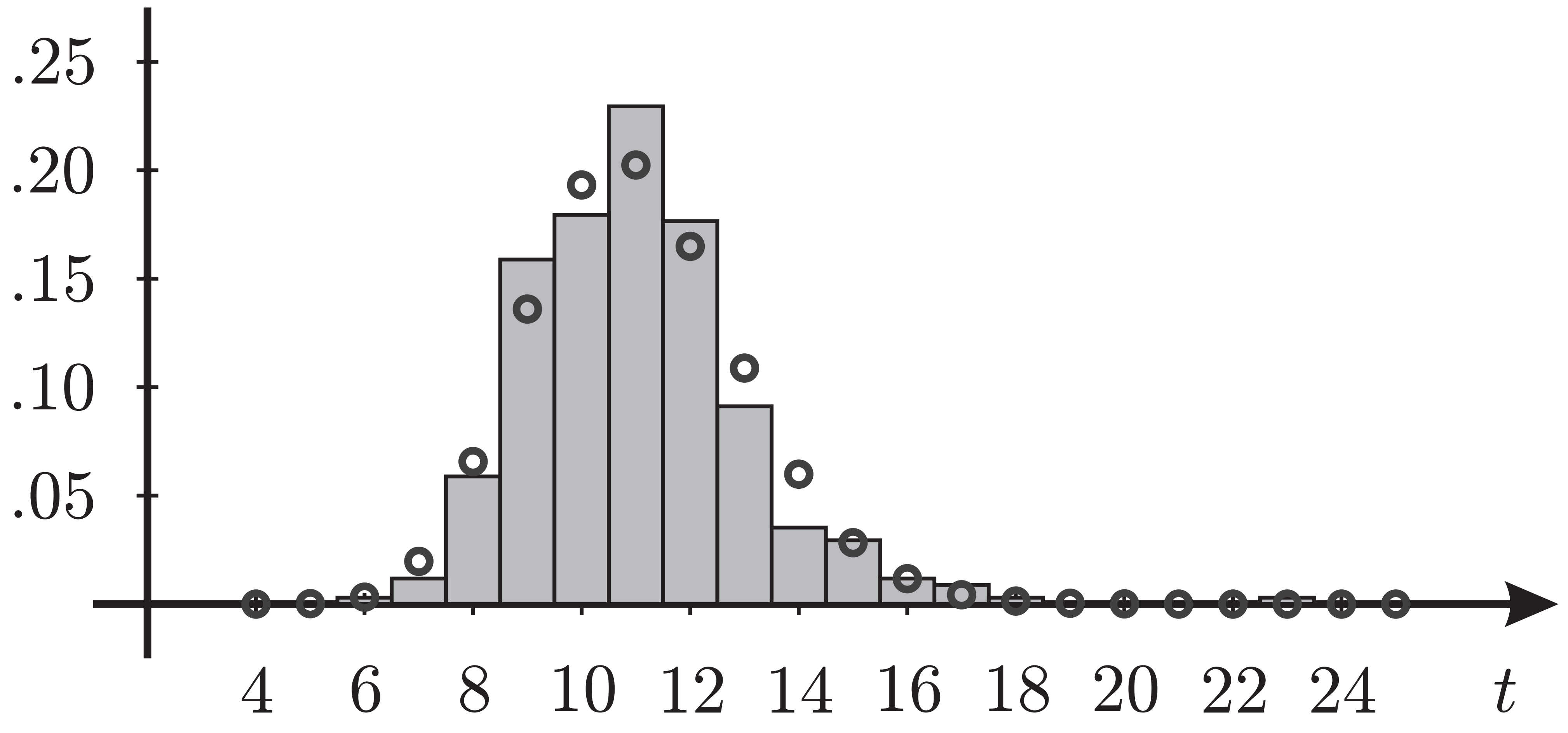
  E.g., $P(X \geq 4) = P(\{\omega \in \Omega \mid X(\omega) \geq 4\})$

- Random variables strictly generalize the way we can talk about probabilities
  - lets us be specific about any transformations
  - switches language from events to boolean expressions

- From this point onwards, we will exclusively reason in terms of random variables

Example: Histograms

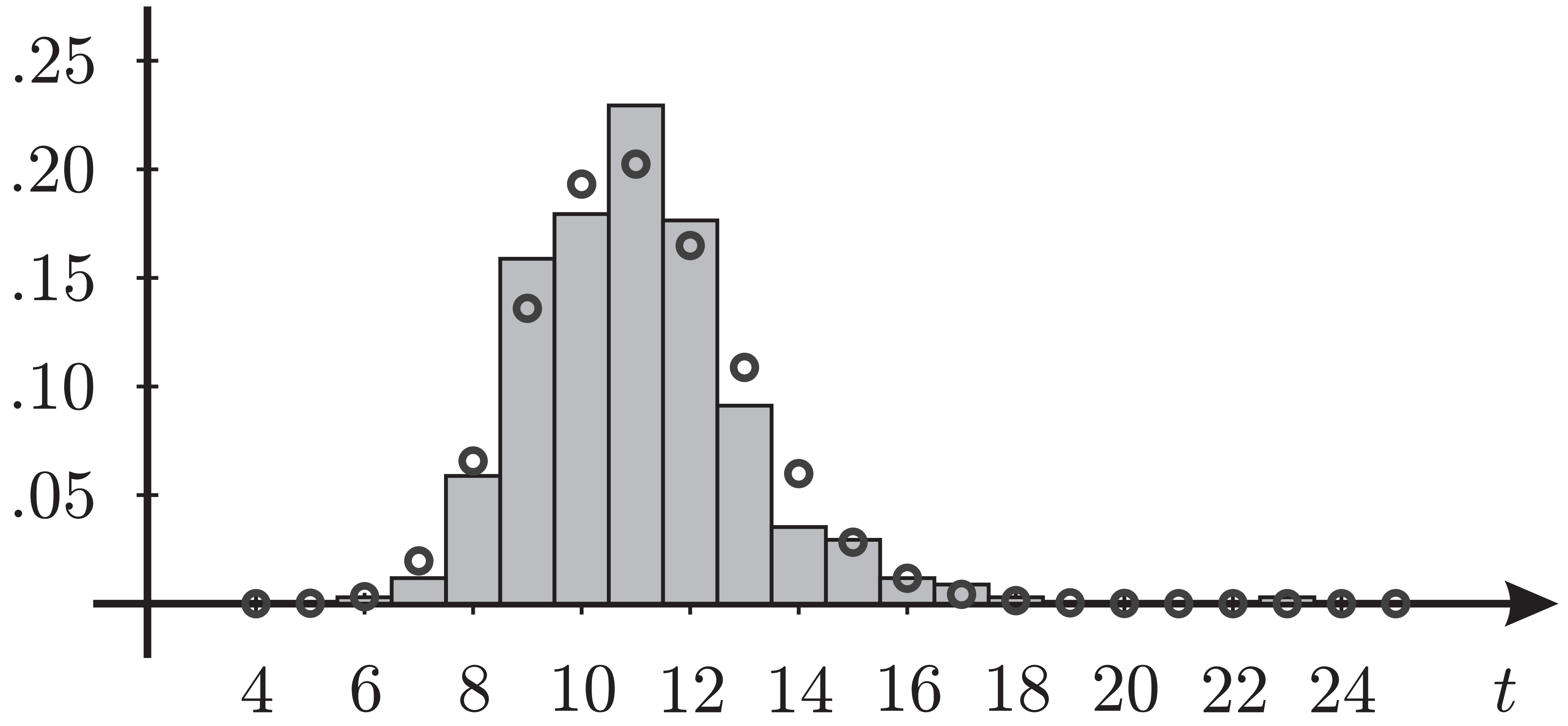
Consider the continuous commuting example again, with observations 12.345 minutes, 11.78213 minutes, etc.



- **Question:** What is the random variable? X = commute times (continuous RV)

Example: Histograms

Consider continuous commuting example, with observations 12.345 mins, 11.78213 mins, etc.



- **Question:** What is the random variable? X = commute times (continuous RV)
- **Question:** In what sense is X a transformation? We were already talking about commute times.

  Answer 1: It is not really a transformation, just a renaming to allow for Boolean expressions

  Answer 2: It was always an RV (even though we didn't call it that), since it is a function of underlying outcomes and events (dynamics in the world)

Summary

- Probabilities are a means of **quantifying uncertainty**
- A probability distribution is defined on a measurable space consisting of a **sample space** and an **event space**.
- **Discrete** sample spaces (and random variables) are defined in terms of **probability mass functions** (PMFs)
- **Continuous** sample spaces (and random variables) are defined in terms of **probability density functions** (PDFs)
- **Random variables** let us reason about probabilistic questions at a more abstract level